Julia Robinson Mathematics Festival

Objective

In the land of Königsberg, everything is made up of one, continuous line. Students will trace images to determine if something belongs in Königsberg.

Rules:
1. Start at any vertex (dot).
2. Try to trace the whole picture without lifting your pencil.
3. You can only trace each edge (line) once.

Introduction

Tell the students that Königsberg is an old puzzle, almost 300 years old. It began with an actual city named Königsberg that was crisscrossed by seven bridges. A famous mathematician named Leonhard Euler wanted to find a way to walk through the city so that he would cross each of those bridges once and only once and today they're going to figure out if Euler could do that. But before that, you want them to figure out what else belongs in the city of Königsberg where everything can be traced without ever crossing the same line more than once.

Explain
Have the students offer suggestions for how to trace the square. Pretend to misunderstand and make some purposeful mistakes (such as going over the same line twice or connecting two vertices that aren't joined by a dotted line) in order to reinforce the rules.

Engage
Ask a student to try tracing the second image (house). Encourage them to explain their thinking out loud as they trace.
If their trace leads to a position where it's impossible to complete, have them explain why it's impossible. Encourage them to begin again.

Common misconceptions
Students might think that:
a. They have to start at A.
b. They can connect two vertices that aren't joined by a dotted line.
c. They aren't allowed to go through the same *point* more than once.

Exploration

In pairs, have your students explore the rest of the puzzles by tracing a path on the plastic sheet protector using a dry erase marker.

Circulate and ask questions to encourage deeper thinking:

a. Which tasks did you find the hardest so far? The easiest? Why?
b. Have you found a strategy that works for all or many of the tasks?
c. When a student is stuck, ask:
    i. Can you try a different starting point?
    ii. What do you know so far?
    iii. What didn't work?
    iv. What are you thinking about trying?
d. To support pattern recognition, ask:
    i. For each puzzle, circle your start and end points. What do you notice?
    ii. Which puzzles were impossible? What do you think made them impossible?
    iii. Can you predict whether a puzzle is possible before solving it? How do you know?
    iv. In this puzzle [*choose an impossible puzzle*], what could you change to make it possible?
e. "Tell me more." is a great basic prompt for getting a student to explain their thinking.

Extend

1. Have your students draw images for the questions on Extension sheet #1.
2. Have student create their own tracing puzzles for others to try.
3. Have students try to solve Euler's original 7-bridges problem on Extension sheet #2.

Discussion

As a group, have students share something about their experience with Königsberg. Try to have at least 3 students share out. Variations of the questions asked earlier are great for generating discussion, such as:

a. Do you have a strategy that worked for more than one puzzle?
b. Why do you think you can solve puzzles 2 and 5 but not puzzle 6?
c. How do you know if a puzzle is impossible?

Materials

1. Königsberg tasks sheets pp. 7-12
2. Dry erase sleeves
3. Dry erase markers

Optional: Königsberg instructions sheet p. 6
Königsberg extensions sheets pp. 13-14

To explore the activity yourself, you can try our digital version here:
jrmf.org/activities/konigsberg

Assessment

Evidence of student learning during problem-solving activities can be obtained from three sources: observations, conversations, and products.

Observation involves actually observing students while they perform tasks and demonstrate skills and may take the form of a checklist or quick note.

Conversation involves engaging students in discussion that encourages them to articulate what they are thinking and then capturing that with a quick note.

Products are student-created records that capture not only their answer, but some of the process that led them to the answer.

Standards

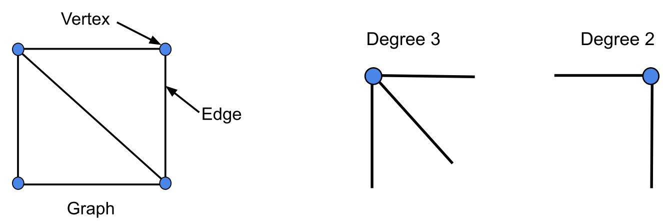
1. Make sense of problems and persevere in solving them.
   CCSS.MP1
2. Construct viable arguments and critique the reasoning of others.
   CCSS.MP3
3. Model with mathematics.
   CCSS.MP4
4. Look for and make use of structure.
   CCSS.MP7

Answers

General Answers:

Note: This is for your information only. Although some students might notice that whether a puzzle is solvable is somehow related to the vertices, they are not expected to know/learn the language.

In school, the term "graph" usually refers to the *x* and *y* coordinates on a Cartesian plane. In graph theory, a graph is any set of points with lines connecting some of them. A point is called a vertex and a line connecting two vertices is called an edge. The number of edges connected to a vertex is called the degree of that vertex.



1. If an image has no vertices with an odd degree (i.e. a point with an odd number of lines connected to it), it is possible to trace starting at any vertex.
2. If an image has two vertices with an odd degree, it is only possible to trace starting at one of these two vertices.
3. If an image has any other number of vertices with an odd degree, it is impossible to trace.

Possible Student Answers to Questions 1 to 9:

1. Solve puzzles 1, 2, 3, 4
   - Puzzle 1 is solvable starting at any point and always ends back at the starting point.
   - Puzzle 2 is solvable starting at A or C.
   - Puzzle 3 is solvable starting at B or C.
   - Puzzle 4 is solvable starting at A or B.

2. One of these puzzles is possible and the other is impossible. Which puzzle is possible? Write out how you would solve it. Which puzzle is impossible? Can you find a path that traces as much of the picture as possible?
   - Puzzle 5 is possible starting at B or E.
   - Puzzle 6 is impossible, e.g., The closest I got on Puzzle 6 is: C to D to G to C to B to A to F to E to D.

3a. Derrick solved puzzle 7 by starting at c and ending at c. Can you solve puzzle 7 starting at a different point? How many different points can you start at and still solve puzzle 7?
   - Solvable from any point.

3b. Derrick solved puzzle 8 by starting at a and ending at g. Can you solve puzzle 8 starting at a different point? How many different points can you start at and still solve puzzle 8?

- Puzzle 8 is solvable only when starting at A or G.

4. Try puzzles 9 and 10. For each puzzle, how many different points can you start at and solve the puzzle?
   - Puzzle 9 is solvable from any starting point. Puzzle 10 is impossible.

5. Try puzzles 11, 12, and 13. You can solve one of these puzzles by starting and ending at the same point. Which puzzle is it? What makes this puzzle special? Why do you think you can't solve the other puzzles by starting and ending at the same point?
   - Puzzle 12 is solvable starting and ending at the same point because it doesn't have any odd-degree vertices (*adult answer*).

6. Erin made 2 three-story towers that aren't in the app. One of these two towers is impossible to trace. Make a prediction: Which tower do you think is impossible to trace? Why?
   - Tower 2 is impossible because it has more than 2 odd-degree vertices (*adult answer*).

General Answers to Extension questions:

1. Answers will vary.

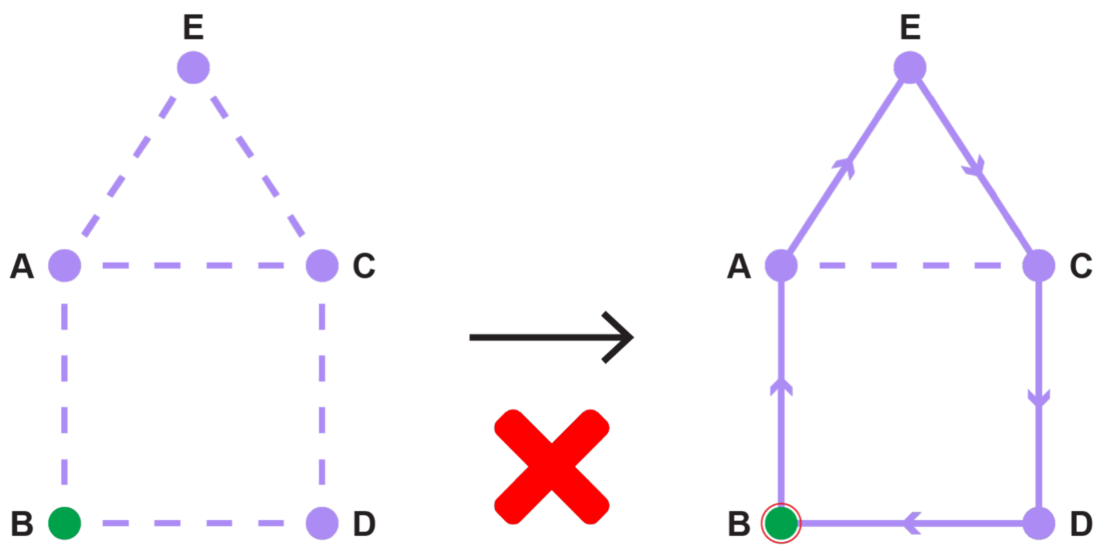
2. There is no solution as all four of the land masses in the original problem (A, B, C, and D) are touched by an odd number of bridges (one is touched by 5 bridges, and each of the other three is touched by 3). Graphs for which it is possible to trace a Euler path have at most two vertices of odd degree.

Königsberg Instructions

In the land of Königsberg, everything is made up of one, continuous line. Here's how to figure out if something belongs in Königsberg.

Steps:

- Start at any vertex (dot).
- Try to trace the whole picture without lifting your pencil.
- You can only trace each edge (line) once.



This trace of the house that started at the green dot didn't work.
Can you start at a different vertex and find a way to trace the entire house without lifting your pencil?

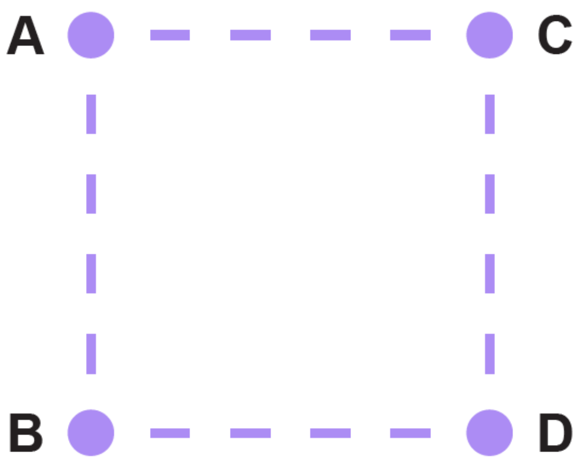
Now try tracing the rest of the puzzles.
Which ones belong in Königsberg?

Königsberg Tasks

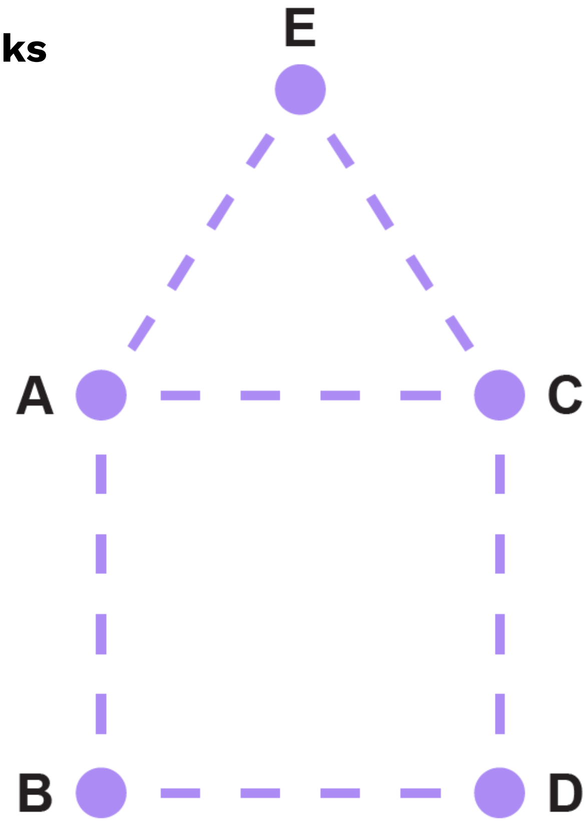
Question 1

The first puzzle is for practice.

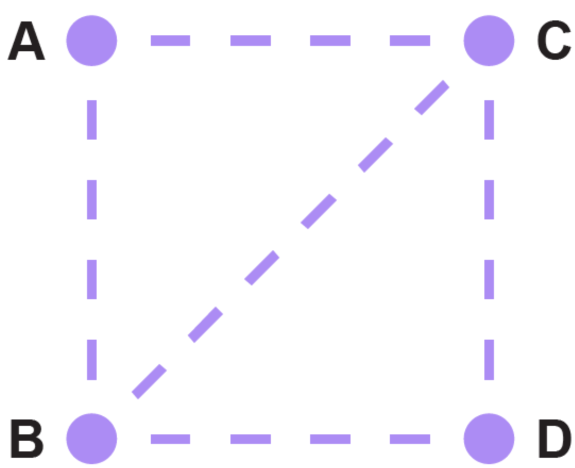
Now try solving the next three puzzles.



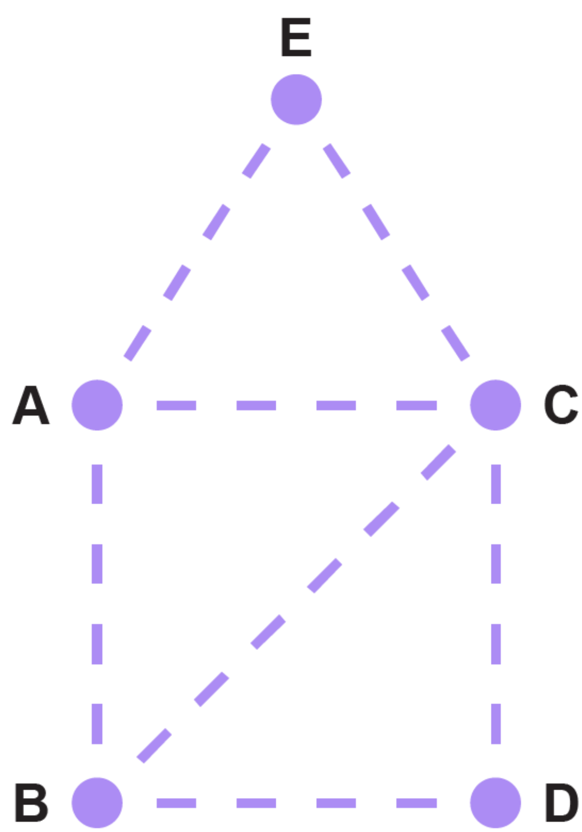
Puzzle 1



Puzzle 2



Puzzle 3



Puzzle 4

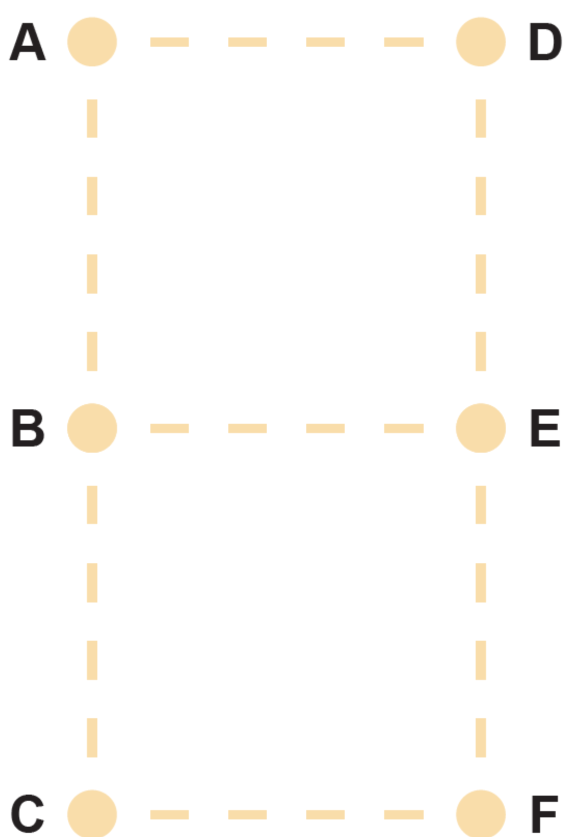
Question 2

One of these puzzles is possible and the other is impossible.

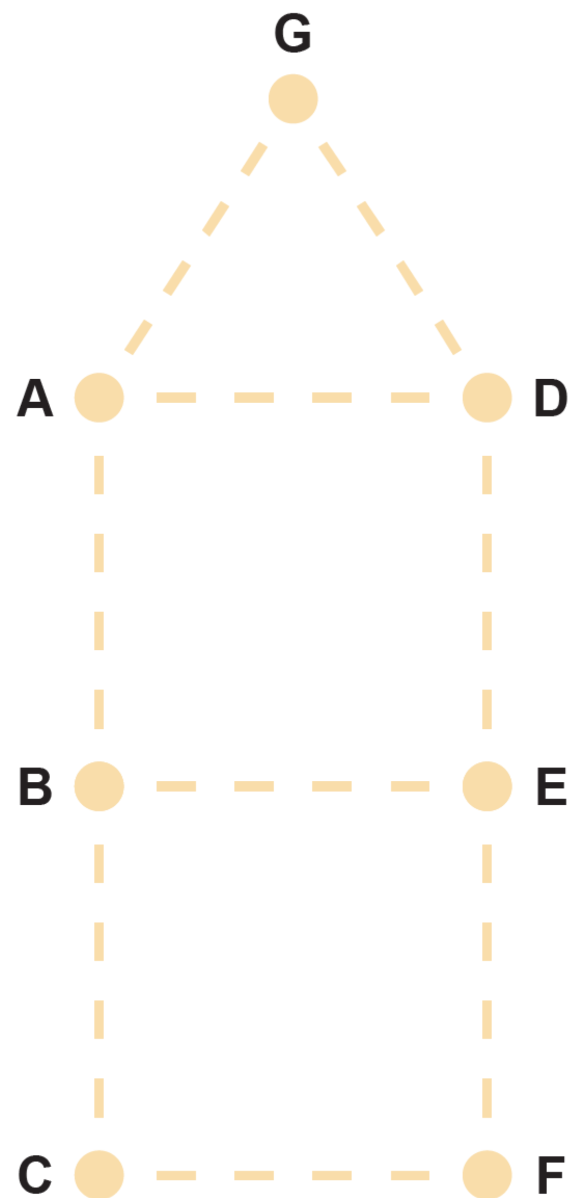
Which puzzle is possible?

Which puzzle is impossible?

Can you find a path that traces as much of the picture as possible?



Puzzle 5



Puzzle 6

Question 3a

Derrick solved puzzle 7

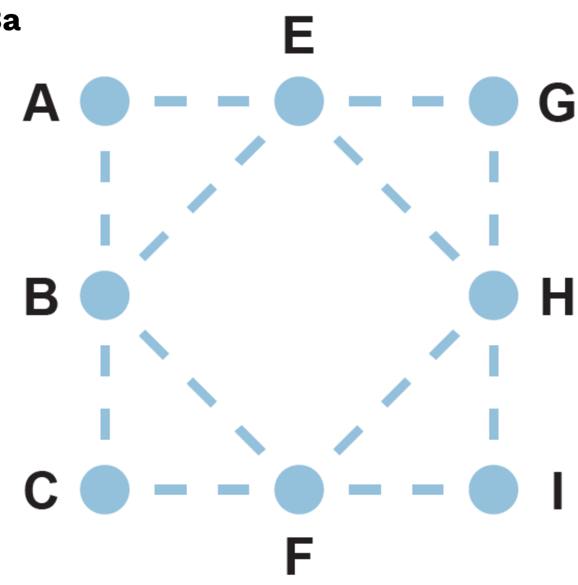
by starting at c and ending at c.

Can you solve puzzle 7 starting

at a different point?

How many different points can you

start at and still solve puzzle 7?



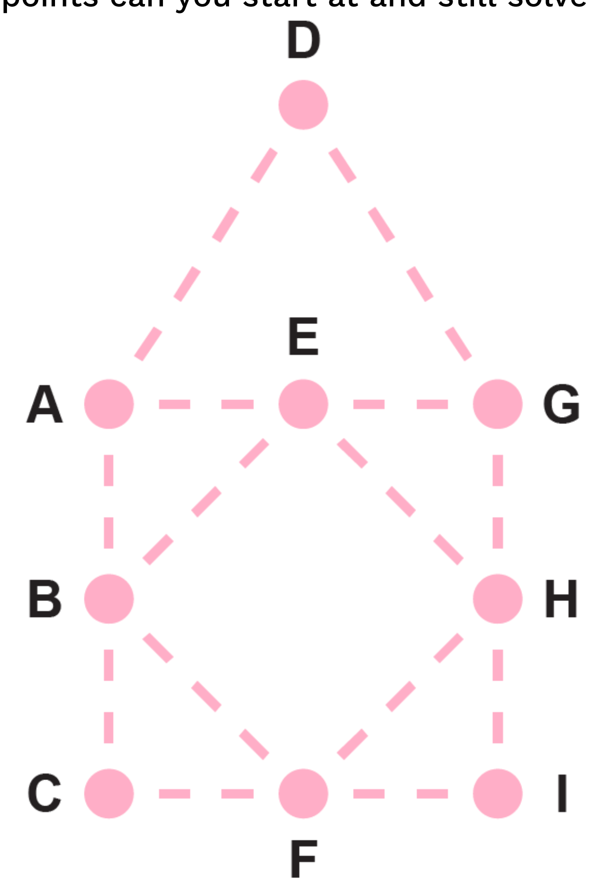
Puzzle 7

Question 3b

Derrick solved puzzle 8 by starting at a and ending at g.

Can you solve puzzle 8 starting at a different point?

How many different points can you start at and still solve puzzle 8?

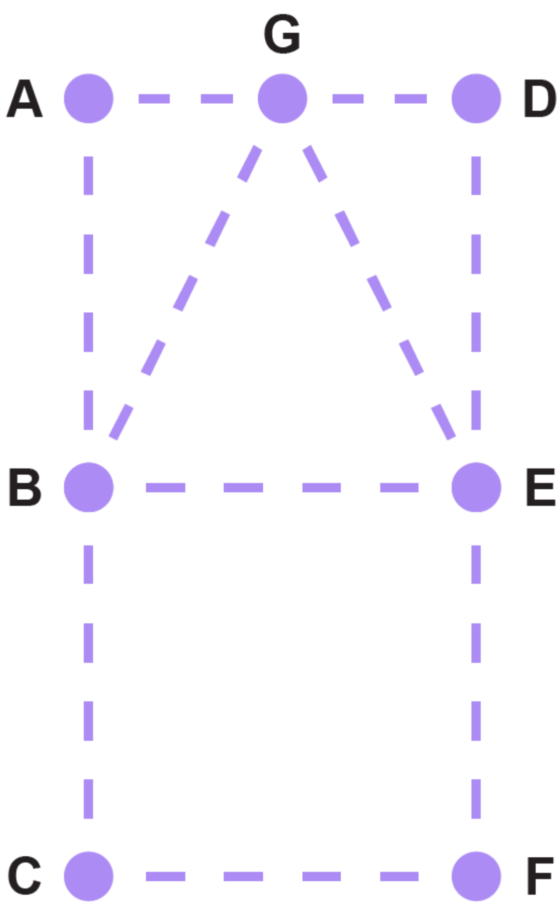


Puzzle 8

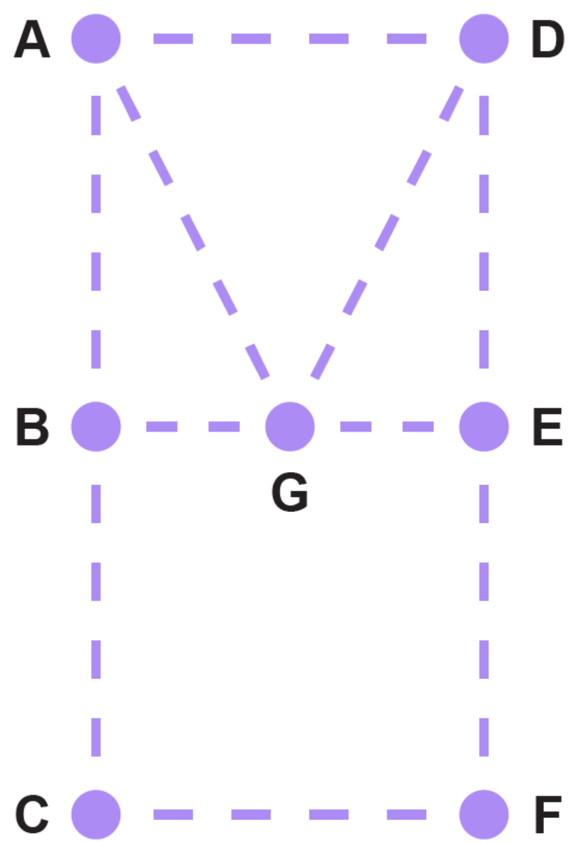
Question 4

Try puzzles 9 and 10.

For each puzzle, how many different points can you start at and solve the puzzle?



Puzzle 9



Puzzle 10

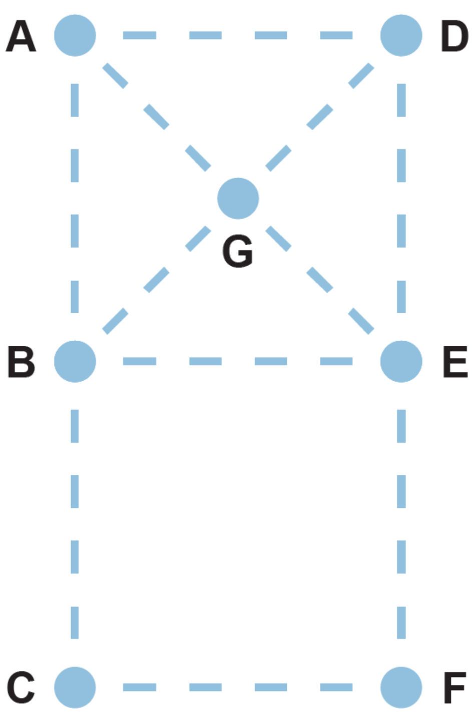
Question 5

Try puzzles 11, 12, and 13.

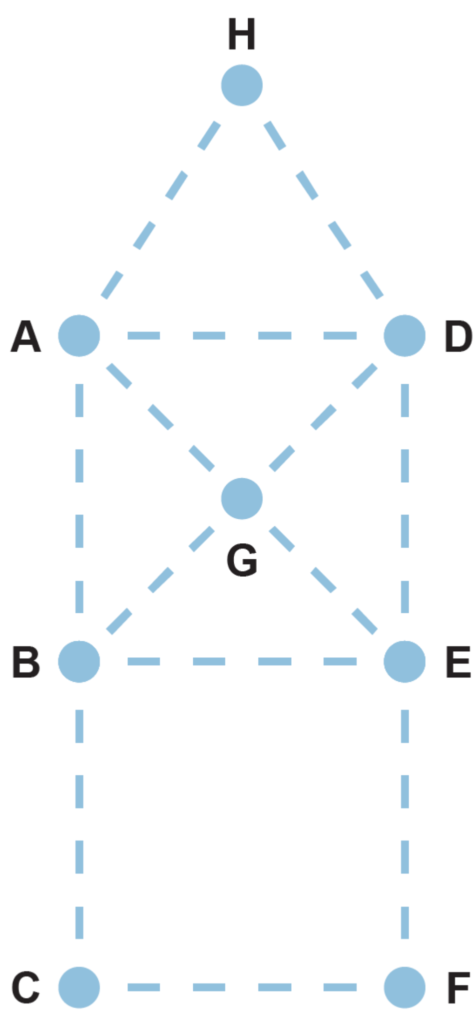
You can solve one of these puzzles by starting and ending at the same point. Which puzzle is it?

What makes this puzzle special?

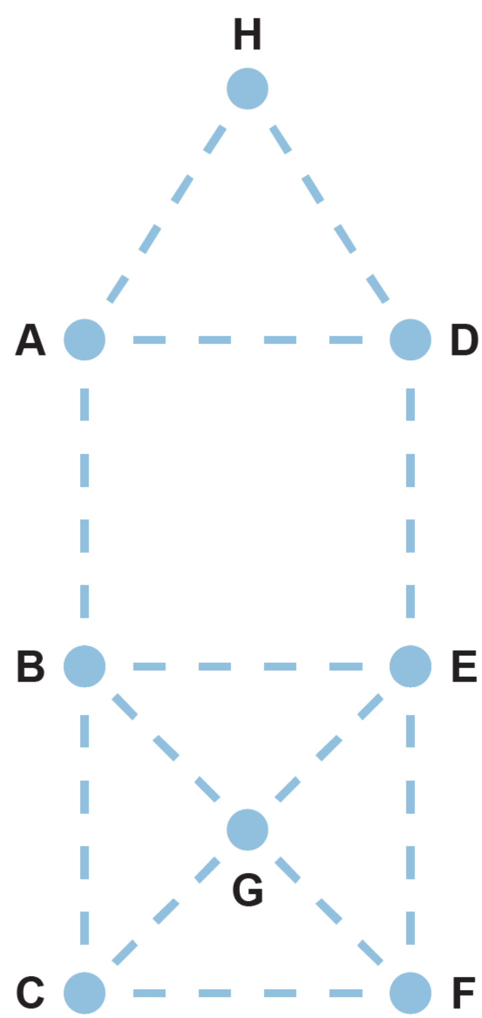
Why do you think you can't solve the other puzzles by starting and ending at the same point?



Puzzle 11



Puzzle 12

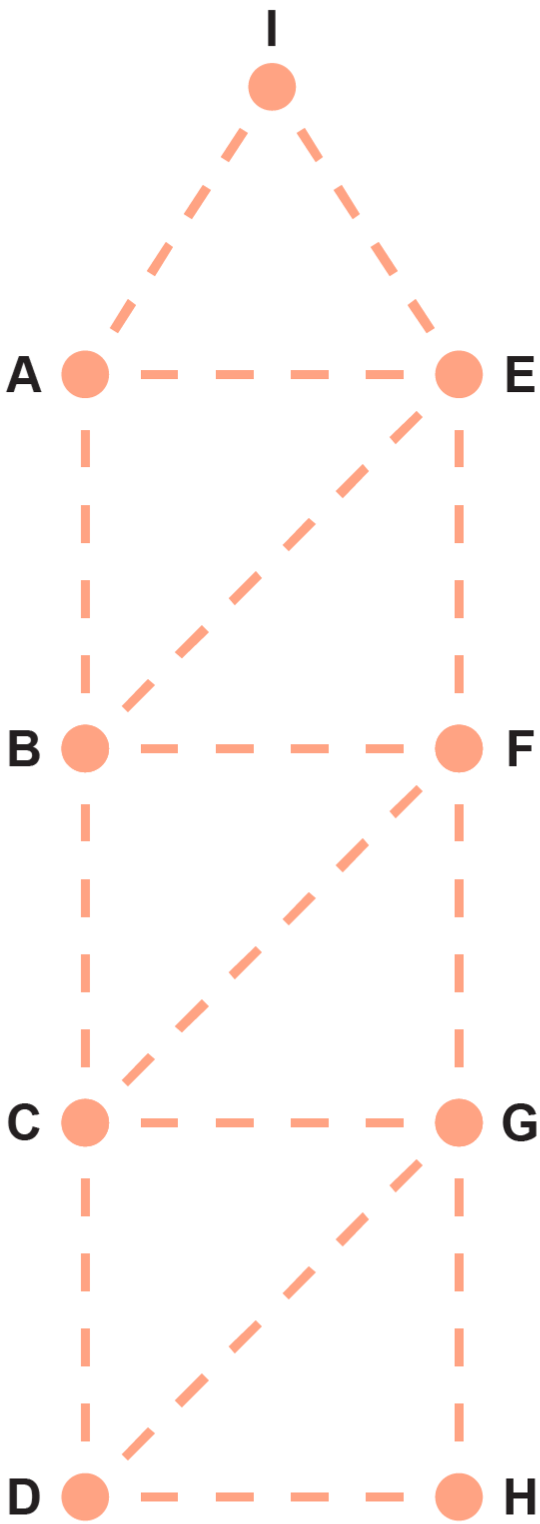


Puzzle 13

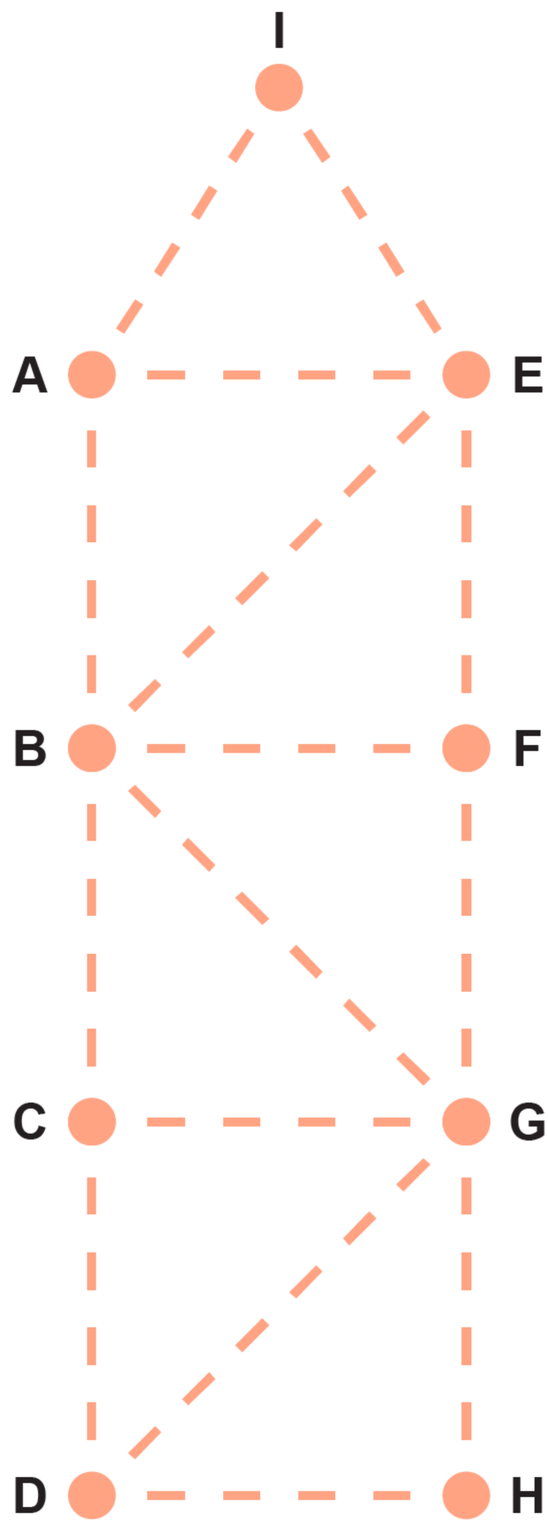
Question 6

One of these three-story towers is impossible to trace.

Make a prediction: Which tower do you think is impossible to trace? Why?



Tower 1



Tower 2

Königsberg Extension #1

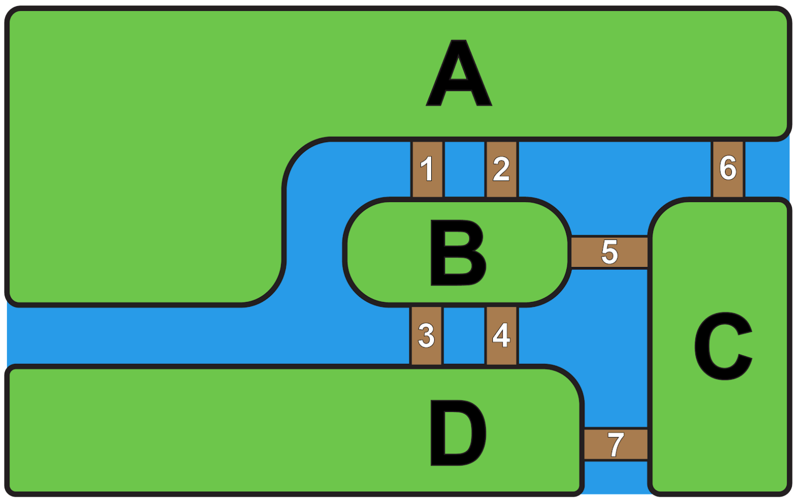
It's your turn to draw some new items for the people of Königsberg. Try to draw a different item for each of the situations below.

1. Draw something with 6 vertices that does not belong in Königsberg.
2. Draw something with 12 edges that does belong in Königsberg.
3. Draw something with 8 vertices that can be traced in more than two different ways.
4. Draw something with 10 vertices and 15 edges that belongs in Königsberg.
5. Draw something that does not belong in Königsberg and has the smallest possible number of vertices and edges.
6. Draw something that does not belong in Königsberg unless you remove one of its edges.
7. Draw something that does not belong in Königsberg and still won't belong even if you remove any one of its edges.
8. Come up with your own Königsberg drawing challenge!

Königsberg Extension #2

Königsberg was a real city, and is now known as Kaliningrad, Russia.
In 1736, mathematician Leonhard Euler tried to walk over each of the seven bridges in Königsberg exactly once.

1. **Can you find a way to cross each of the seven bridges exactly once?**
   If so, show how. If not, explain why you can't. You do not have to go in numerical order.



2. Let's pretend that some of the bridges have been moved around in Königsberg. For each of the images below, can you find a way to cross each bridge exactly once? If so, can you find more than one way to do so?