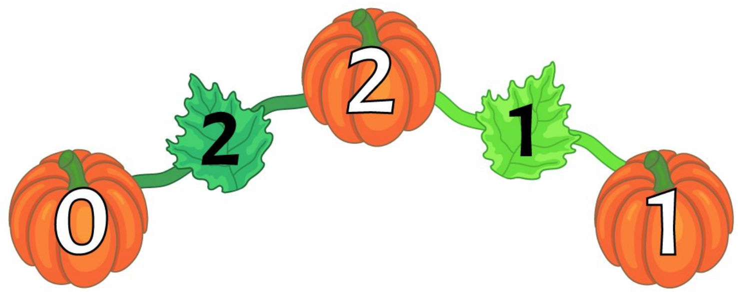
Graceful Gourds



Julia Robinson Mathematics Festival

App

jrmf.org

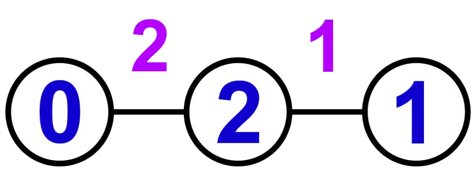
Graceful Graphs

Objective:

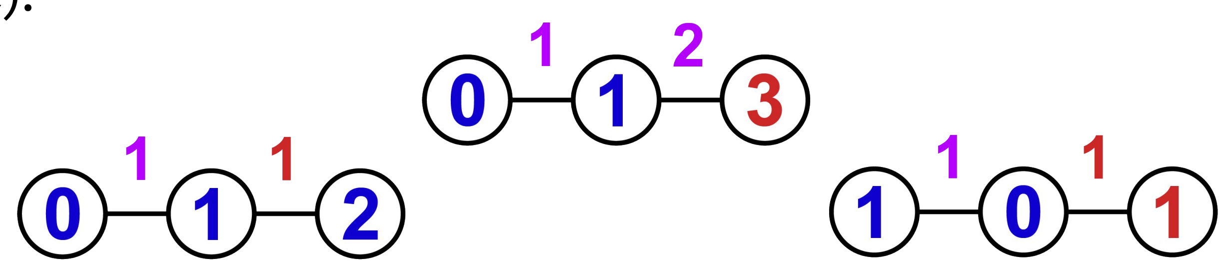
For each path, find a graceful labeling or explain why one can't exist.

Rules:

- Labeling numbers start at 0. For your path, count the number of edges (lines between circles) to decide the last labeling number. For example, the path below has 2 edges, so our labeling numbers will be 0 through 2.
- Place the labeling numbers into the vertices (circles) using each no more than once and then label each edge with the difference of the numbers in the two vertices it joins.
- The graph has been gracefully labeled when the edges are labeled with consecutive counting numbers starting from 1 up through the number of edges (2 in our example).



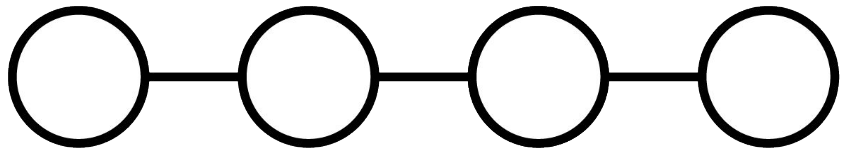
Graceful labeling of a 3-path



Ungraceful labelings of a 3-path

Paths

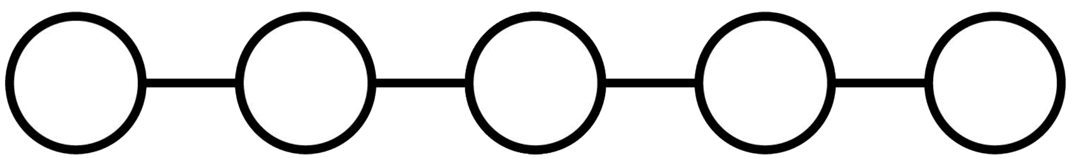
1. Find a graceful labeling of the 4-path below. (Label the vertices 0-3 so the edges are labeled 1-3 by their differences.)



2. Is there another graceful labeling of this path?

3. Are there any numbers that have to go next to each other?

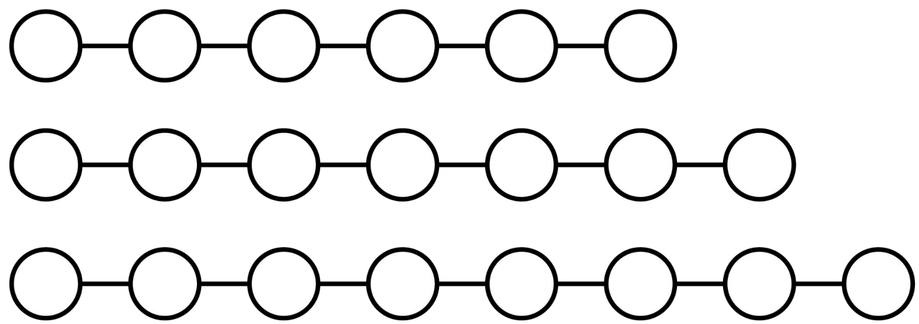
4. Can you find a graceful labeling of the 5-path below? (Label the vertices 0-4 so the edges are labeled 1-4.)



5. How many graceful labelings of this path can you find?

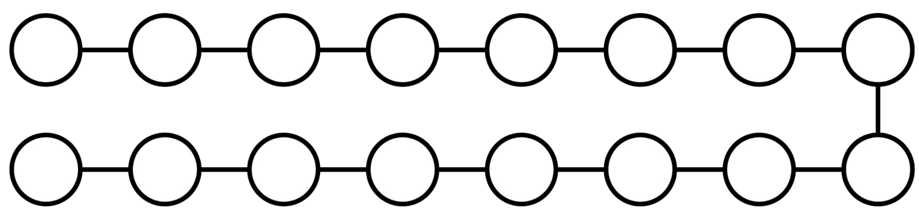
Longer Paths

1. See if you can find graceful labelings for the following paths.



2. Do you have a system for gracefully labeling paths? Can you describe to a friend how it works?

3. Try your system on a longer graph, like the one below!



4. Is it possible to end with the edges labeled 1 through m in that order?

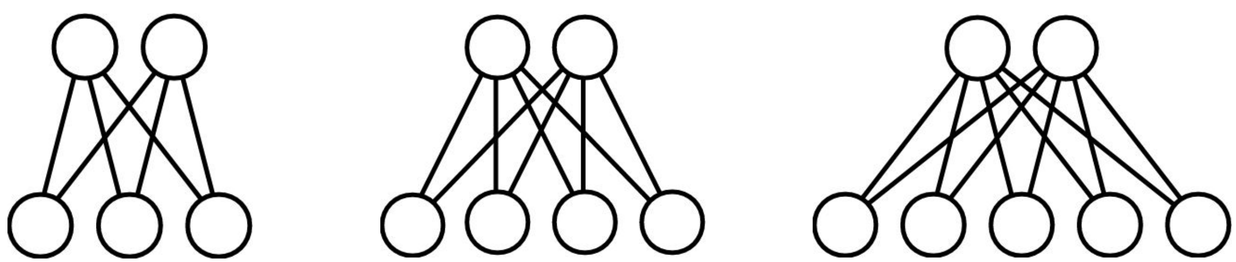
Utility Graphs

In a utility (or complete bipartite) graph, a row of houses and a row of utility companies are set up so that each house is connected to each utility. If there are a houses and b utilities, the total number of edges is $m = a \cdot b$ and you will label your vertices with your choice of the numbers

$$0, 1, 2, \ldots, a \cdot b$$

(You won't use all of these numbers -- you get to choose which!) Again, your goal is to end with the edges labeled 1 through m.

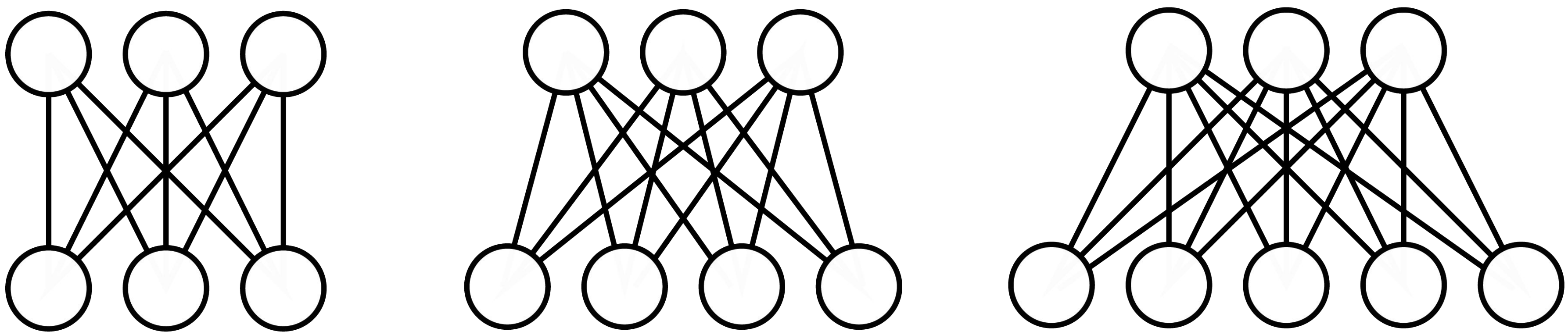
1. Can you find a graceful labeling for the utility graphs below?



2. Can you see a way to always gracefully label graphs of this family no matter how long the bottom row gets if the top row has 2 vertices?

Greater Utility

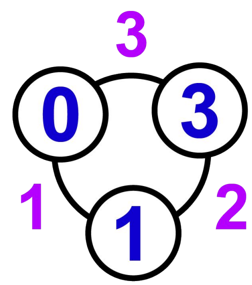
1. What if we increase the number of vertices in the top row to 3?



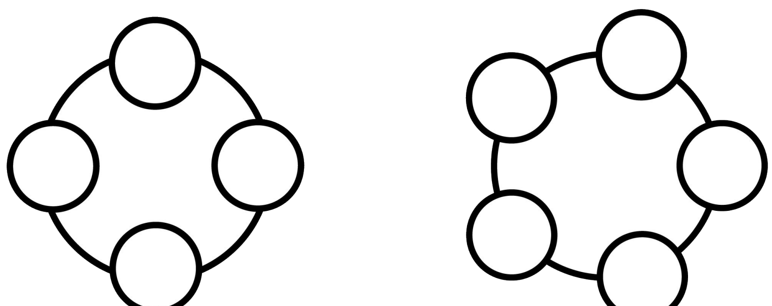
2. Can you describe a general method for gracefully labeling utility graphs no matter how many vertices are in each row?
3. How would your numbering system change if a random edge were deleted? Two random edges?

Cycles

A cycle has the same number of edges and vertices. To gracefully label the m edges of a cycle with 1 through m you will label the vertices with all but one of the numbers 0 through m. (You choose which number not to use!) Below is a graceful labeling of a 3-cycle. Can you find others?



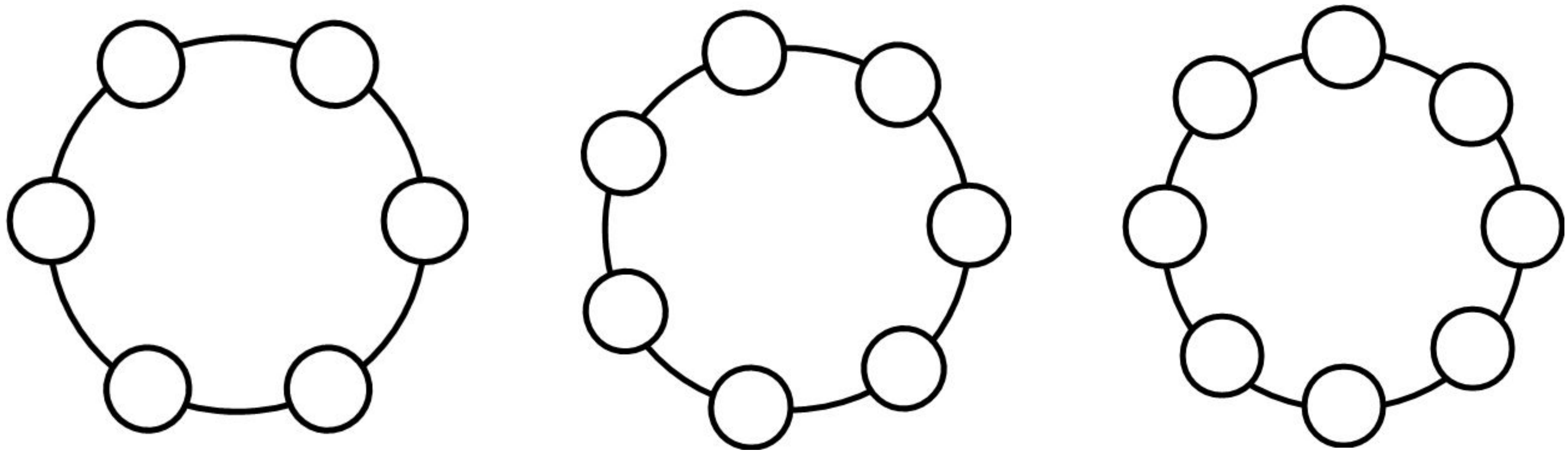
1. One of the cycles below can be gracefully labeled and the other cannot. Which is which?



2. Are there any numbers you have to use when labeling a cycle?

Bigger Cycles

1. Explore bigger cycles, like the ones below. Which can be gracefully labeled and which cannot?



2. Is there any pattern to when a cycle can be gracefully labeled or not?

3. For cycles that can be gracefully labeled, do you have a system for labeling them? Can you explain it to a friend?

4. For cycles that can't be gracefully labeled, can you show why they can't be?

Other Graph Families

More generally, a graceful labeling of a graph with m edges and n vertices requires labeling the vertices with n numbers chosen from

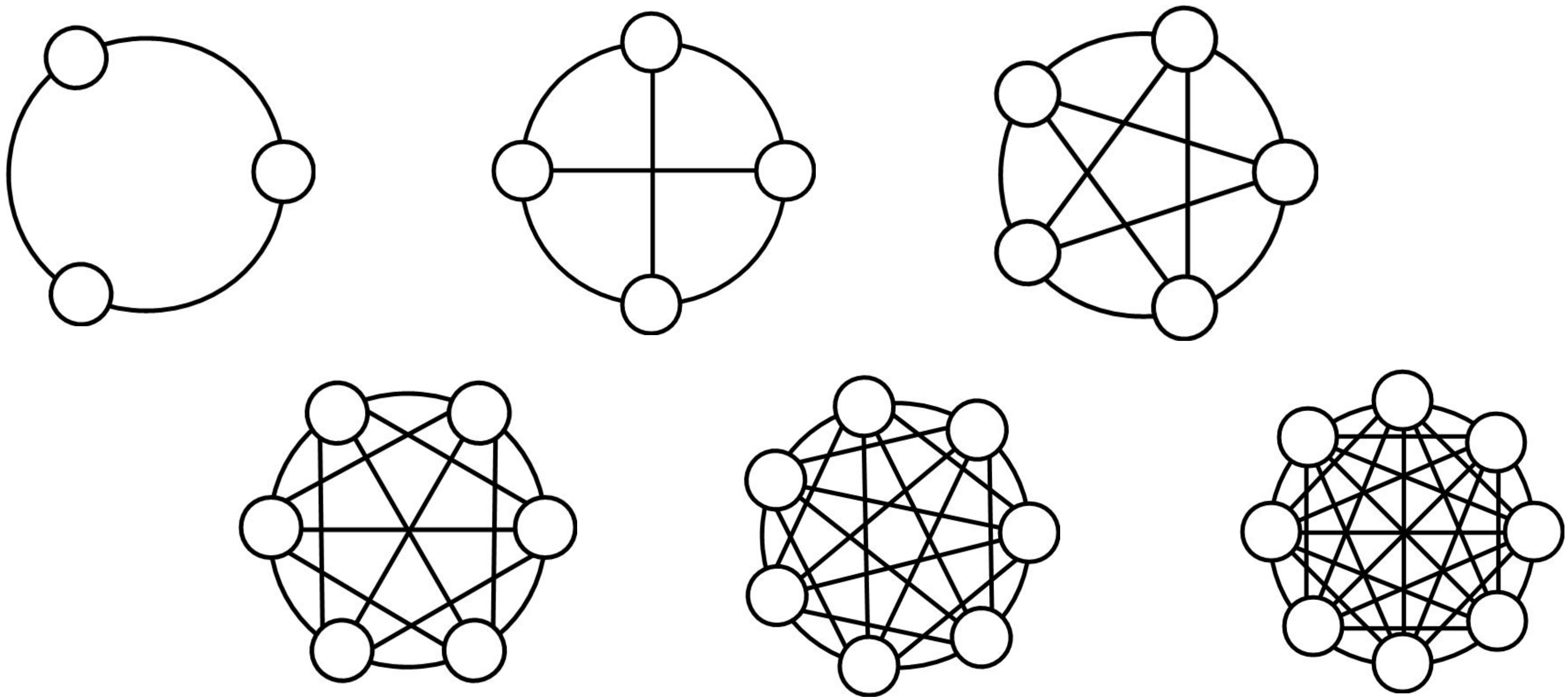
$$\{0, 1, 2, \ldots, m\}$$

so that each of these numbers is used at most once and the differences of the vertex labels make it so the edges labels are 1 through m. (You get to choose which vertex numbers to use!)

In the following slides, a selection of graphs from different graph families are presented for you to explore.

Complete Graphs

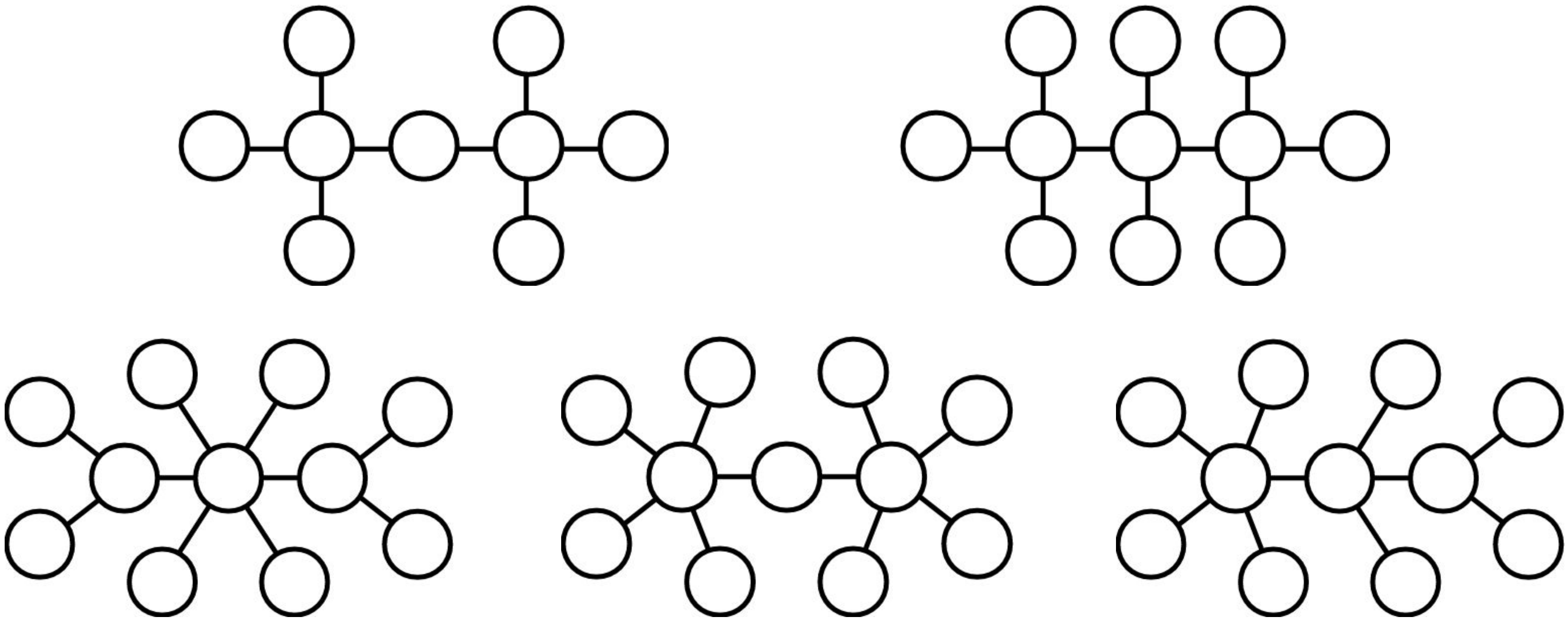
Which of the following complete graphs have graceful labelings? (A graph is complete if each vertex is adjacent to every other vertex.)



Can you find a system for gracefully labeling a complete graph when such a labeling exists? For showing why such a labeling can't exist when it doesn't?

Caterpillars

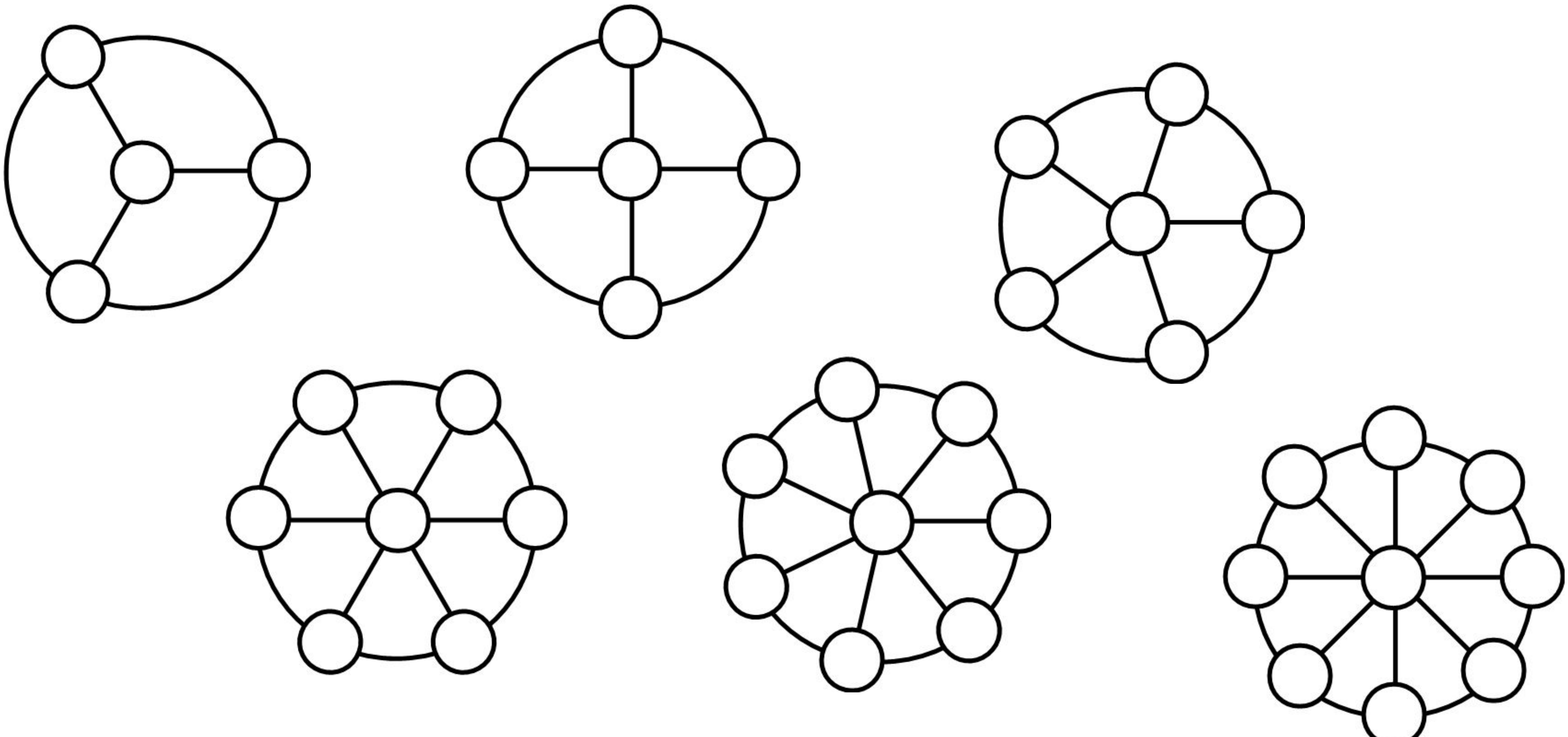
Which of the following caterpillar graphs (paths with “fuzz”) have graceful labelings?



Can you find a system for gracefully labeling a caterpillar graph when such a labeling exists? For showing why such a labeling can’t exist when it doesn’t?

Wheels

Which of the following wheels (cycles with an added axle and spokes) have graceful labelings?



Can you find a system for gracefully labeling a wheel when such a labeling exists? For showing why one can't exist when it doesn't?

Big Trees

A tree is a graph that contains no cycles (like the ones we've seen) as subgraphs. Mathematicians have verified with computers that trees with 35 vertices or fewer are graceful, but we still don't know much beyond that! Try your hand at creating graceful labelings for the following trees: