Julia Robinson Mathematics Festival

Countdown Teaching Guide

Objective

The goal is to take the last token.

Rules:
1. Start with 10 tokens in a pile.
2. Players take turns taking 1 or 2 tokens.

Introduction

Count out a pile of 10 tokens and challenge a student to see who can take the last token… of course, there are rules!

Explain
1. Tell the students that you'll each take turns. On each turn, you can choose to take one or two tokens.
2. The person who takes the last token wins.

Engage
1. Play a game against a student (or pair of students).
2. Encourage them to explain their thinking out loud as they choose which move to make.
3. Encourage them to take suggestions from the rest of the class.

Common misconceptions
Students might think that:
1. They can take less than the exact amount on the final move.
   For example, if there are only 2 tokens left and you can take 1 or 3, they might think that by taking 3 they can take the 2 that remain and win.

Exploration

In pairs, have the students explore the game, starting with a pile of 10 tokens.

Circulate and ask questions to encourage deeper thinking:
a. If they win: Do you think you can win again? Can you come up with a strategy so that you win every time?
b. What if you had more tokens? Less tokens?
c. How can you use your knowledge of smaller games to help you create strategies for larger games?
d. Near the end of a game, how soon can you see who will win? How do you know?

e. If the other player has a winning strategy but doesn't make a winning move, can you always win?
f. What are the numbers of tokens that you like to see left on your turn? What are the numbers of tokens that you don't like to see?
g. When a child is stuck, ask:
    i. Can you try playing with a smaller pile of tokens?
    ii. What have you tried so far?
    iii. What worked? What didn't work?
    iv. What are you thinking about trying?
h. To support pattern recognition, ask:
    i. What if you track your winning and losing positions?
    ii. Is there a good starting move?
    iii. What do you think will happen next?
i. "Tell me more." is a great basic prompt for getting a child to explain their thinking.

Extend
Have the student pairs try the challenges on pages 7 to 8.

Discussion

As a whole group, have students share something about their experience with Countdown. Try to have at least 3 students share out. Variations of the questions asked earlier are great for generating discussion, such as:

a. Were you able to come up with a strategy where you win every time?
b. What are the numbers of tokens that you like to see left on your turn? What are the numbers of tokens that you don't like to see?
c. Is anyone sure they can beat me?

Materials

About 20 counters or other small objects per pair of students..

Optional: Countdown instructions and tasks p. 6
Countdown extensions pp. 7-8

To explore the activity yourself, you can try our digital version here: jrmf.org/puzzle/countdown

Assessment

Evidence of student learning during problem-solving activities can be obtained from three sources: observations, conversations, and products.

Observation involves actually observing students while they perform tasks and demonstrate skills and may take the form of a checklist or quick note.

Conversation involves engaging students in discussion that encourages them to articulate what they are thinking and then capturing that with a quick note.

Products are student-created records that capture not only their answer, but some of the process that led them to the answer.

Standards

1. Make sense of problems and persevere in solving them.
   CCSS.MP1
2. Construct viable arguments and critique the reasoning of others.
   CCSS.MP3
3. Model with mathematics.
   CCSS.MP4
4. Look for and make use of structure.
   CCSS.MP7

Answers

Take 1 or 2 tokens:

The player with the winning strategy depends entirely on the number of tokens in the pile. If playing without any errors,

- Winning strategy: Always leave the pile a multiple of 3.
- Player 2 has the advantage if the pile starts a number of tokens that is a multiple of 3. Player 1 has the advantage otherwise.

Take 1, 2, 3, ..., n-1, or n tokens:

- Winning strategy: Always leave the number of tokens in the pile a multiple of n+1.
- If the starting number of tokens is a multiple of n+1, then the second player can deploy this strategy. Otherwise, the first player has the advantage.

More generally:

- You can think of each starting number of tokens as being winning positions or losing positions depending on whether the player who must make a play has the advantage or not. For example, if the player may take 1, 3, or 6 tokens, then it's clear that piles of size 1, 3, or 6 are winning positions, since the player who must make a move can win the game by taking all tokens. A pile of size 0 is a losing position, because the player who would move next has just lost.
- For other positions, here's a way to decide whether they are winning or losing:
  - If a move can be made to turn it into a losing position, then it is a winning position (W).
  - If all possible moves lead to winning positions, then it is a losing position (L).

We can decide which positions are which by making a table. Using the example of the game where players may take 1, 3, or 6 tokens, we can start the table:

Tokens	0	1	2	3	4	5	6	7	8	9	10	11	12
W or L	L	W		W		W							

If there are 2 tokens, the only amount that can be taken is 1, which leads to a W, so 2 tokens is an L. For 4 tokens, 1 or 3 can be taken, but both lead to Ws, so 4 tokens is an L. For 5 tokens, by taking 1 token we then arrive at an L, so 5 tokens is a W. We can fill the table out for as many tokens as we want:

Tokens	0	1	2	3	4	5	6	7	8	9	10	11	12
W or L	L	W	L	W	L	W	W	W	W	L	W	L	W

This table provides a winning strategy for one of the players:

- If the number of tokens on your turn is a W, then take any amount that will turn it into an L. We know there's such an amount, since that's how the table was made.
- If the number of tokens on your turn is an L, then the other player will win if playing perfectly, so make a random move and hope they mess up!

These tables will always eventually start to repeat themselves, but it can be tricky to tell when the repetition will start and what the length of the repeated block will be.

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This activity was made in collaboration with sfmathcircle.org

Countdown Instructions and Tasks

How to play:

Take turns taking either **1 token or 2 tokens** from a pile.

Whoever takes the last token wins!

1. **Start with 10 tokens and play a few games.**
   a. Can you find a strategy that helps you win every time?

2. **Does your strategy work with more tokens? Fewer tokens?**

Countdown Challenges

1. Take 1, 2, or 3 tokens

Play a few games with a 10-token pile, but this time players may take 1, 2, or 3 tokens.

- Which player seems to have the advantage?
- What does the pile look like when you know you're about to win or lose?
- What if you start with 11 tokens? 12 tokens? *n* tokens?

Make a table to track your findings and see if you can explain the pattern you see.

2. Take 1, 2, 3, or more tokens

Play a few games with a 10-token pile, but this time players may take 1, 2, 3, or more tokens.

- Can you find a winning strategy for Player 1 or Player 2?
- Is there a pattern you can generalize?
    - Choose a number *k*. How does this game work if players are allowed to take 1, 2, 3, ..., or *k* tokens from a pile?
    - If there is an *n*-token pile, for which *n* does Player 1 have a winning strategy and for which *n* does Player 2 have one?

Make a table to track your findings and see if you can explain the patterns you see.

3. Take 1, 2, or 4 tokens

Now try playing the game with a 10-token pile but each player can take 1, 2, or 4 tokens -- but not 3 tokens!

- Can you find a winning strategy for Player 1 or Player 2?
- Does the winning strategy change hands for 11 tokens? 12? 13? 14?
- Which player has the winning strategy and what is it on an n-token pile?
- Are the patterns you notice similar to patterns in the previous tasks? Why?

Countdown Challenges

4. Take 1, 3, or 4 tokens

Now try playing the game with a 10-token pile but each player can take 1, 3, or 4 tokens -- but not 2 tokens!

- Can you find a winning strategy for Player 1 or Player 2?
- Does the winning strategy change hands for 11 tokens? 12? 13? 14?
- Which player has the winning strategy and what is it on an n-token pile?
- You might notice a pattern to who has the winning strategy that looks a lot like one of the patterns you saw earlier with consecutive takes. Can you explain why these patterns are so similar?

5. Hot takes

Let's try designing our own game! Choose some numbers that players are allowed to take. For example, {1,3,6} or {1,2,4,8} could work. If your set doesn't include 1, then it might be the case that the last token cannot be taken, in which case the last player to make a move wins.

- For which numbers n of starting tokens does Player 1 or Player 2 have the advantage?
- You might notice that some of the patterns you see start off a little weird before settling into something more predictable. Can you find a way to use your numbers to predict when the pattern will repeat? Can you use them to predict what the eventual repeating pattern will be? ({1,6,9} and {3,7,8} are interesting to explore!)

6. Hotter takes

What if you are allowed to take two numbers m and n?

- Can you say anything generally without choosing two specific numbers? (For example, what if the numbers are 1 and n? What if both numbers are odd?)
- Can you say anything about who has the winning strategy for which pile sizes? Maybe you can't predict the exact pattern, but you might be able to predict how long it goes before it repeats itself! (This length is called its period.)

Mathematicians don't have a good way to predict what will happen when the subtraction set has three or more numbers in it, except in a few special cases. Try some games with a subtraction set of the form {a,b,c} picking interesting numbers for a, b, and c. What can you say about it?