Teeter-totter Geometry and Adding Areas

Notation: AB will be used to represent both the segment AB and the length of line segment AB. When it is necessary to avoid ambiguity $|AB|$ will be used to represent the length. $[ABC]$ represents the area of $\triangle ABC$.

1. **Equal Altitude Property.** If two triangles have equal altitudes then the ratio of their areas is the same as the ratio of their bases.

   (a) Find the ratio of $[ABD] : [CBD]$, if D is the midpoint of AC.

   (b) Find the ratio of $[ABD] : [CBD]$, if $AD : DC = 5 : 6$.

   (c) Find the ratio of $[ABC] : [AFC]$ where $\overleftrightarrow{BF} \parallel AC$.

2. In $\triangle ABC$ with E on AB and D on BC, AD intersects CE at P, $AE : EB = 4 : 3$ and $BD : DC = 6 : 1$.

   (a) What is $[AEP] : [BEP]$ and $[CDP] : [BDP]$?

   (b) Let $[AEP] = aX$, $[BEP] = bX$, $[CDP] = cY$, $[BDP] = dY$, and $[APC] = Z$. Find simple integer values for $a, b, c,$ and d.

   (c) Notice that $\dfrac{[AEC]}{[BEC]} = \dfrac{4}{3} = \dfrac{aX + Z}{bX + (c+d)Y}$. Solve for Z in terms of Y.

   (d) Notice that $DP : PA = cY : Z$. What is the value of this ratio?

   (e) In $\triangle ADB$ and $\triangle ADC$, apply the same Area Addition technique used above in parts (c) and (d) to find $EP : PC$.

If the algebra in the previous problem was too complicated for you, don't panic. We will now investigate a simpler way to approach the problem. The fundamental tool will be the principle of the lever or the

Teeter-totter principle. Two masses on a seesaw will balance if the product of the one mass and its distance from the fulcrum (balancing point) is equal to the product of the other mass and its distance from the fulcrum.

3. If a mass of 12 g is 5 m from the fulcrum and there is a mass 4 m from the fulcrum on the other side which balances the teeter-totter, then how many grams does that mass weigh?

We will now consider a collection of objects of the form aA consisting of a positive real number a and a point A in the plane. Define addition of these objects as follows: $aA+bB=cC$ where $c=a+b$ and C is the point on segment AB that balances the masses at A and B (i.e., $a|AC|=b|CB|$).

It can be shown that addition defined this way is associative, so that $aA+(bB+cC)=(aA+bB)+cC$ and the resulting mass point with a mass of $a+b+c$ is located at the unique physical center of mass (balancing point) of aA, bB, and cC.

4. Now reconsider the problem where $AE:EB=4:3$ and $BD:DC=6:1$. To balance AB at E, assign 4 to A and 3 to B, giving $3A+4B=7E$. Also assign $4\cdot 6/1=24$ to C to balance BC at D, giving $4B+24C=28D$. Now $3A+(4B+24C)=3A+28D=31P$ (i.e., P balances AD).

   (a) What is the ratio $DP:PA$? (Compare with your previous answer in Problem 2.)

   (b) What is the ratio $EP:PC$? (Compare with your previous answer in Problem 2.)

   When the masses assigned to each vertex are equal then the balancing point will be the actual center of mass.

5. Use mass point geometry to find the ratio in which the medians are divided by the center of mass. That's right; all of the medians are divided in the same ratio.

Mass point geometry generalizes to higher dimensions. Assuming a unique center of mass for solids and the associative law:

6. Show that in any tetrahedron (a pyramid with four triangular faces) every line segment from a vertex to the center of mass of the opposite face is divided by the center of mass of the tetrahedron in the same ratio. What is that ratio?

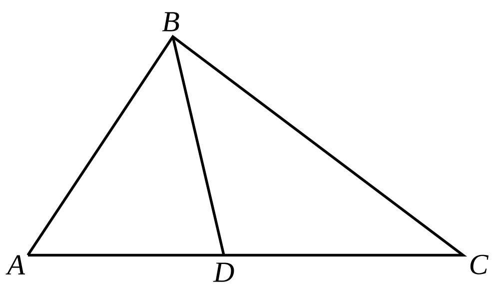
7. In $\triangle ABC$, D, E, and F are the trisection points of AB, BC, and CA nearer to A, B, and C, respectively.

   (a) If $BF\cap AE=J$, show that $BJ:JF=3:4$ and $AJ:JE=6:1$.

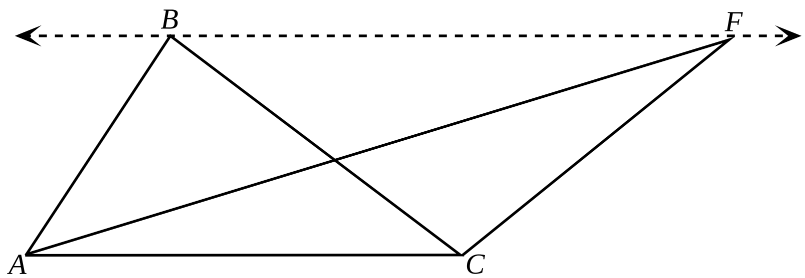
   (b) Let $CD\cap AE=K$ and $CD\cap BF=L$. Extend part (a) to show that $DK:KL:LC=1:3:3=EJ:JK:KA=FL:LJ:JB$.

   (c) Use parts (a) and (b) to show that the area of $\triangle JKL$ is $\frac{1}{7}$ the area of $\triangle ABC$.

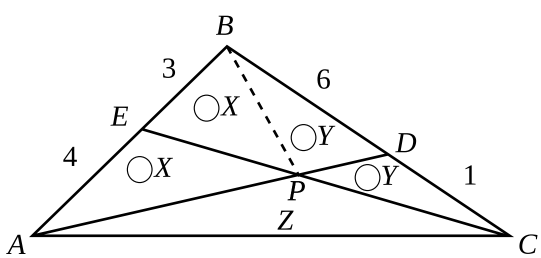
8. (AIME '85 #6) In $\triangle ABC$, cevians AD, BE and CF intersect at point P. The areas of $\triangle$'s PAF, PFB, PBD and PCE are 40, 30, 35 and 84, respectively. Find the area of triangle ABC.



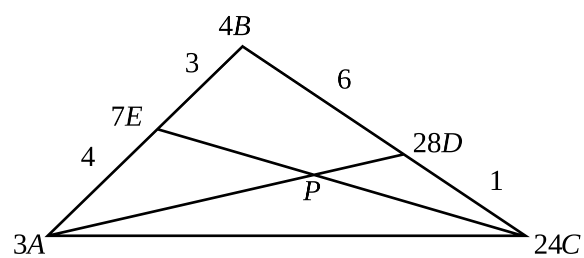
Problem 1(a)–(b)



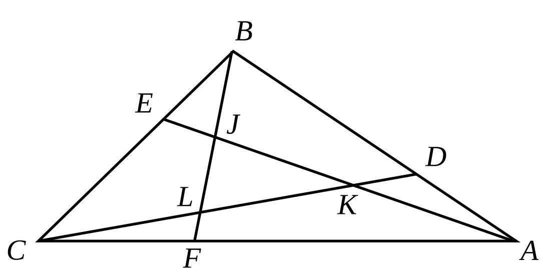
Problem 1(c)



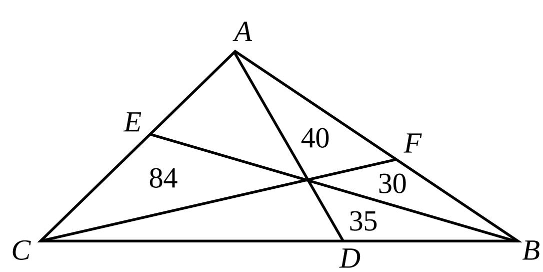
Problem 2



Problem 4



Problem 7



Problem 8