Pythagoras Revisited

The usual Pythagorean theorem says that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse. Or, equivalently, if a triangle is a right triangle, then the sum of the squares of the sides adjacent to the right angle is equal to the square of the side opposite the right angle.

Another way to describe the theorem: The square of the diagonal of a rectangle is equal to the sum of the squares of two of its sides. Or, equivalently, the sum of the squares of the diagonals is equal to the sum of the squares of all four sides.

1. Prove that this holds for a parallelogram, too: the sum of the squares of the diagonals is equal to the sum of the squares of all four sides.

2. What other quadrilaterals, if any, that are not parallelograms also have this property? In other words, can you prove that if the sum of the squares of the diagonals of a quadrilateral equals the sum of the squares of its sides, then it is a parallelogram? Or, alternatively, can you describe other quadrilaterals that have this sum of squares property but are not parallelograms?

3. The usual Pythagorean theorem is an "if and only if" statement, so it's also true that if the squares add up appropriately, then the triangle is a right triangle. Prove this statement.

4. The British Flag theorem states that in a rectangle ABCD, with a point P inside the rectangle, $PA^2 + PC^2 = PB^2 + PD^2$. Prove this theorem.

5. Investigate what happens if P is on a corner of the rectangle. Conclude that the British Flag theorem implies the Pythagorean theorem.

6. What if P is on an edge of the rectangle? Outside the rectangle?

7. Prove that when a triangle has medians to sides a and b that are perpendicular to each other, then $a^2 + b^2 = 5c^2$. Is the converse also true?

8. In an isosceles triangle with sides of length c, a segment of length a drawn from the vertex cuts the base into pieces of length b and d. Prove that $c^2 = a^2 + bd$.

EXTRA problems – leaders, use these with kids who are finding the above too easy.

9. W. J. Hazard / cut-the-knot.com: Let parallelogram ABCD be inscribed in parallelogram MNPQ. Draw BK // MQ and AS // MN. Let the two intersect in Y. Then the area of ABCD = area of QAYK + area of BNSY. (This is a generalization of Pythagoras: you can prove the Pythagorean theorem using it, in the case where both parallelograms are squares.) Hint: extend some lines, look for things that have equal areas (because they are congruent, or because they have equal base and equal height).

10. Sam Loyd: A triangular lake has a square built on each side, of area 74, 116, and 370 square units. What is the area of the lake?

11. Build a rectangle with its base on the hypotenuse of a right triangle and its other two vertices on the two legs. This leaves three triangles left in the original triangle. They have inradii r_1, r_2, r_3. Prove that when the area of the rectangle is maximized, $r_1^2 + r_2^2 = r_3^2$.