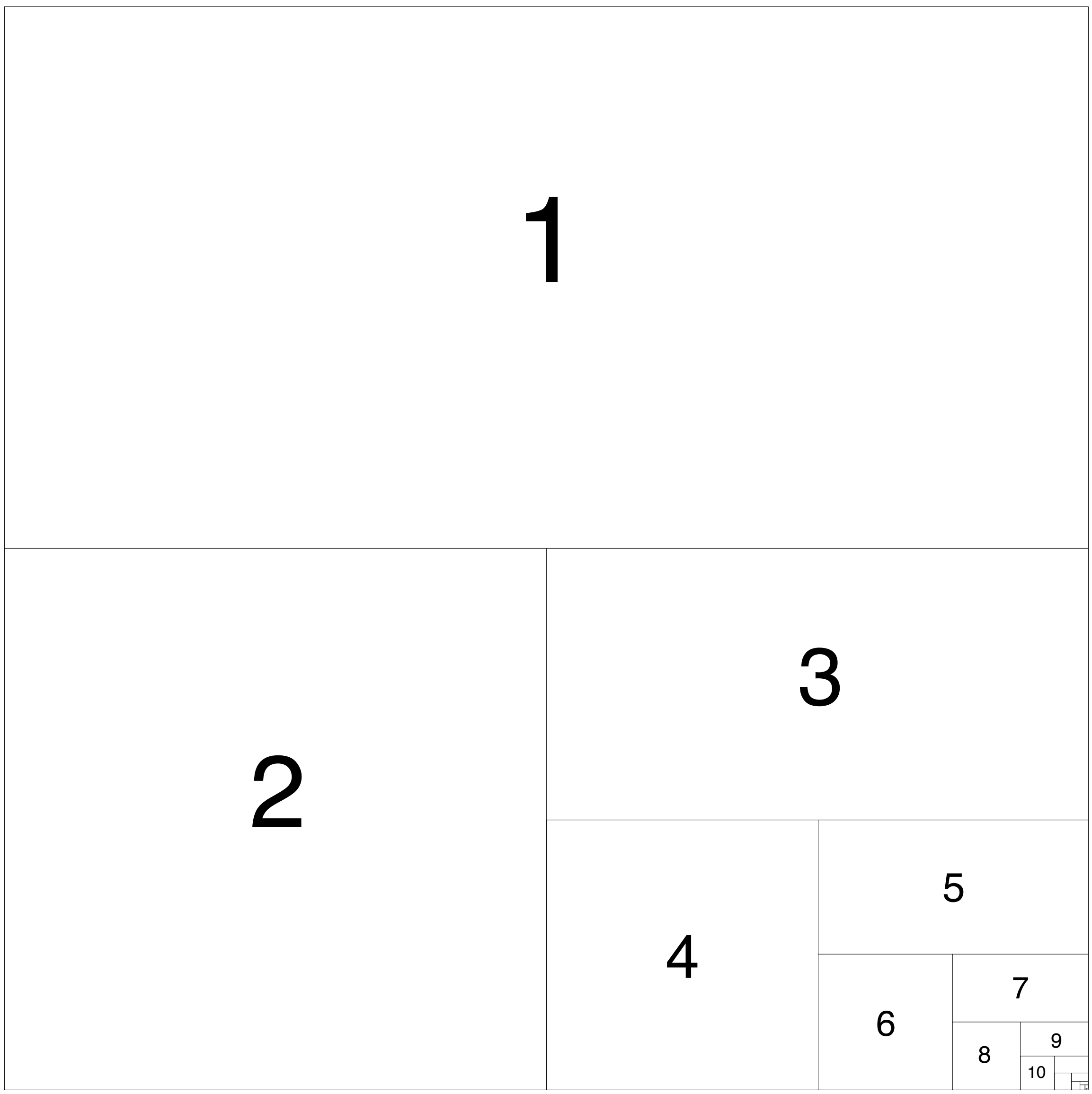
Julia Robinson Mathematics Festival

Directed By Joshua Zucker

# Little Boxes




Imagine that the pattern shown above continues infinitely, so that the boxes in the lower right hand corner keep getting smaller and smaller. Also assume that the entire square is one unit high and one unit wide, so that its area is one.

1. What is the area of the rectangle labeled 1? What is the area of the square labeled 2? How about the rectangle labeled 3? The square labeled 4?... What is the pattern? Can you write down a general expression for the area of the rectangle labeled *n*?

2. What is the sum of the areas of rectangles 1 and 2? What is the sum of the areas of rectangles 1, 2 and 3? How about rectangles 1, 2, 3 and 4? Rectangles 1, 2, 3, 4 and 5?... What is the pattern here? Can you explain it in terms of the picture? Can you write down a general expression for the sum of areas 1 through *n*?

3. If we added up the area of all the rectangles, what would it be? Look at the picture for this one. Then think about your answer to problem 2. How many rectangles would you need to add to get more that 99% of the total area?

4. Wait a minute. The sum of all the areas of the rectangles was a finite number? How can a sum of infinitely many numbers be finite?

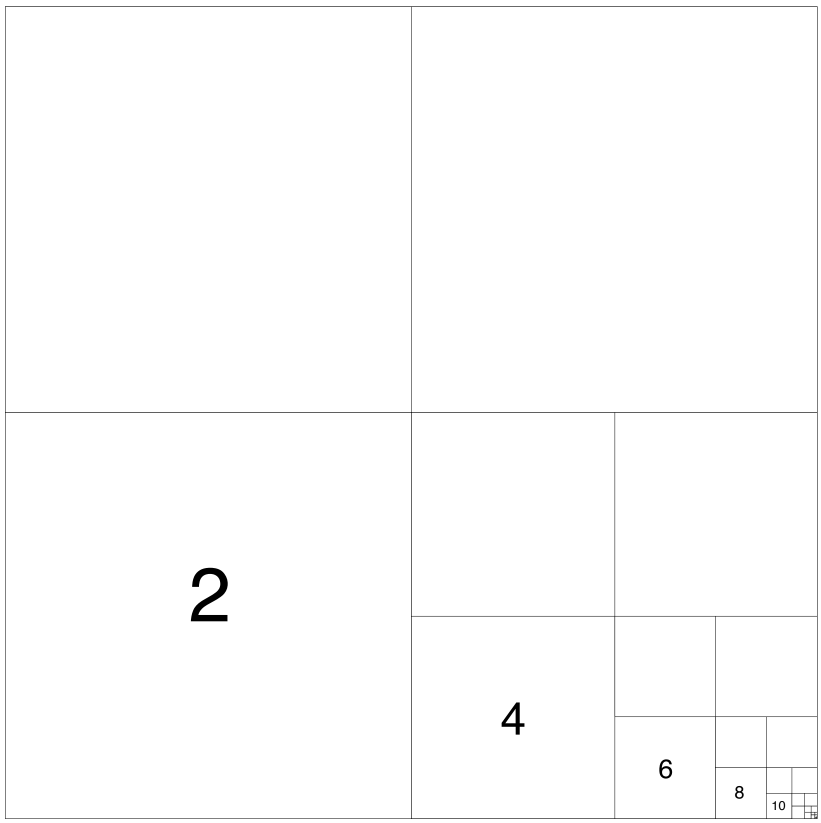
Now let's think about just the even-numbered rectangles in the diagram (the squares).

5. What is the pattern of areas of the squares? Can you write a general expression for the area of the square labeled 2*n*?

6. What is the sum of areas of squares 2 and 4? What is the sum of the areas of squares 2, 4 and 6? How about squares 2, 4, 6 and 8? Squares 2, 4, 6, 8 and 10? What is the pattern here? Can you write down a general expression for the sum of the areas of squares 2 through 2*n*?

7. If we added up the area of all the squares, what would it be? There are two ways to answer this difficult question. One is to try to understand the pattern you found in problem 6, and where it's leading. The other is to think visually. For a visual solution, the picture on the following page might be helpful.

8. What is the sum of the areas of all the odd-numbered rectangles in the first diagram?



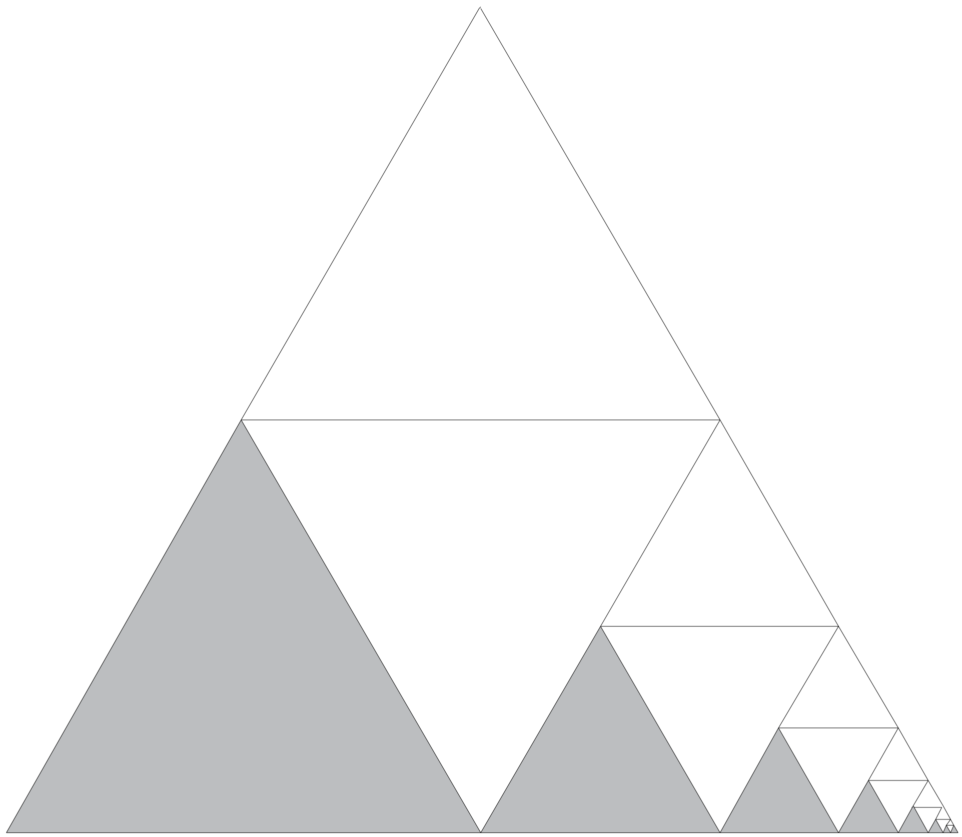

9. What is the sum of the areas of the rectangles labeled 4, 8, 12, 16, 20, etc. (every other square in the first diagram)?

10. What is the sum of the areas of the rectangles labeled 3, 6, 9, 12, 15, etc.?

11. What is the sum of the areas of rectangles labeled with multiples of *n*, for any integer *n*?

12. Can you find a sequence of rectangles whose areas add up to 1/31? How about 3/7? 1/5? 1/9? Is there any fraction between 0 and 1 without a corresponding sequence of rectangles?

13. Assuming that the area of the entire triangle in the diagram below is 1, what is the shaded area?



14. Can you draw pictures for other shapes? Can you divide the square or the triangle into pieces using other ratios?

15. Prove that $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\cdots=1$.

16. Draw a picture of some rectangles that proves it.

17. For each of the above, (squares, rectangles, and your answer to the previous problem) how many different ways are there to make each number between 0 and 1 as the sum of areas in your list?