Grammar Generation

Starting with *S*

We play a game by starting with a string consisting of one letter, S for "start", and following rules that let us produce other strings of letters. The game ends when all the uppercase letters are gone, and you have only lowercase letters remaining. The strings with only lowercase letters are called "ending strings". The set of rules is called the *grammar.*

Game 1: $S \to x$
This rule means that if you have an S, you may turn it into an x.
The grammar has only one rule, and it doesn't get you very far!

1. What can you form by starting with S, using the rules of game 1?

Game 2: $S \to xS$ and $S \to x$
Now we have two rules to choose from. The first rule says, starting with any string containing an S, you can put an x in front of an S. The second rule, as before, says that you can turn an S into an x (to end the game by eliminating the uppercase letter).

One example of the use of these rules goes like this:

$$S \Rightarrow xS \Rightarrow xxS \Rightarrow xxx$$

Note that the single arrow is used in the rules of the game, and the double arrow is used to represent the moves of the game. You might want to number the rules, and put the numbers on top of the double arrows to show which rule is being used.

2. Is it possible to start, as always, with S and end up with $xxxxx$ in game 2? If so, show how. If not, explain why not.

Game 3: $S \to Sx$ and $S \to Sy$ and $S \to x$

3. Describe all the possible strings you can end with in game 3.

4. In game 3a, the rules are $S \to xS$ and $S \to yS$ and $S \to x$.
   In what ways is this the same as game 3? In what ways is it different?

Is the set of possible ending strings the same or different?

Game Design: Odd and Even

5. Design your own game: create rules so that the ending strings can have any odd number of x's, but never an even number of x's.
   a. Write your rules clearly, in the pattern shown for the games above.
   b. Show how to form the string $xxxxxxx$ using your grammar.
   c. Explain why the string $xxxxxx$ cannot be formed using your grammar.

6. Create rules so that the ending strings are exactly those with an even number of x's.
   a. Write your rules clearly, in the pattern shown for the games above.
   b. Show how to form the string $xxxxxx$ using your grammar.
   c. Explain why the string $xxxxx$ cannot be formed using your grammar.

7. Zero is an even number. Does the game you created in the previous problem allow you to create strings with zero x's? If not, perhaps you'd like to use a rule like $S \rightarrow$, which sometimes we write for more clarity as $S \rightarrow \varepsilon$, where the funny greek letter e, pronounced "epsilon", is our symbol for the empty string.
   a. Clearly write rules for the even number of x's game. Make sure that zero x's is possible as well.
   b. Explain how you know that your new rules can make any even number of x's, and no other ending strings.

Game Design: Back and Forth

8. Design a set of rules that can produce exactly those strings beginning with x and alternating x and y thereafter. As usual, clearly write your rules, show how to create a few examples, and also explain why non-examples can't be ending strings. In this case, you might show how to construct $xyxy$, x, and $xyxyx$, and then explain why $xxyx$ cannot be constructed.

9. Design a set of rules for "one y in the middle", with some number of x's followed by one y and then the same number of x's again. For example you can construct xyx and $xxxyxxx$ but not $xxyxxx$. Can your rules construct just y with zero x's in front and back?

10. Design a set of rules that can form any palindromic string of x's and y's. For example, show that you can construct x, y, xyx, xxx, $xyyx$, and $yxxyxxy$ but not $yxxyxx$.

The Capital of ABland, and beyond.
Sometimes we can mix in other capital letters besides the S we start with. Remember, the game ends once all the capital letters are gone.

11. Purity: $S \to A, S \to B, A \to Ax, A \to \varepsilon, B \to By, B \to \varepsilon$
    a. What ending strings can you make using these rules? Hint: Why is the game called Purity?
    b. Explain how you know you can make any string of this type, and only strings of this type.
    c. Explain why we need more capital letters than just S in order to have this result.

12. Design your own: Can you create rules that will result in all, and only, strings with an unequal number of x's and y's? For instance, you should be able to make $xyyxy$ but not $xyyx$ or $xyyxyx$.

13. One way to describe strings with some number of x's followed by an equal number of y's is to say that we want strings of the form $x^a y^a$. Write a set of rules that will produce only strings of the form $x^a y^a$ or give a convincing argument to explain why it can't be done.

14. Write a set of rules to produce only the strings with an equal number of x's and y's in any order.

15. Write a set of rules to produce only the strings of the form $x^a y^{2a}$ or explain why it can't be done.

16. Write a set of rules to produce only the strings of the form $x^a y^a z^a$ or explain why it can't be done.

17. Write a set of rules to produce only the strings of the form *wzw*, where *w* represents the same string of *x*'s and *y*'s each time it is used, or explain why no such set of rules exists.

    For example, your set of rules should be able to produce *xxyzxxy*, *xzx*, and *yyxyzyyxy*, but not *xxyzyxx*.