Julia Robinson Mathematics Festival

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# Fibonacci Flips and Probability Puzzles

There are some methods that are pretty effective at distinguishing a sequence of coin flips generated by a human's imagination from one generated by flipping a fair coin. The idea for one of them is contained in the discussion below.

1. Write down a sequence of 100 H's and T's (or 1's and 0's, or any two symbols you like) to simulate flipping a coin 100 times. Then, as you do the problems below, compare the number of times each thing occurs with how often it should occur according to your probability calculations.

2. Out of all the places where the coin comes up heads, what fraction should be immediately followed by heads? By tails?

3. Out of all possible lists of 100 flips, how many never have more than 1 consecutive flip the same?

4. Out of all possible lists of 100 flips, how many never have more than 2 consecutive flips the same?

5. Why is this activity called "Fibonacci Flips"?

6. For those of you who know matrix multiplication, how could you use it to help solve this problem?

7. Out of all possible lists of 100 flips, how many never have more than 3 consecutive flips the same?

8. Generalize: how would you find the probability of never having 6 consecutive flips the same? (Does your sequence that you invented in problem 1 ever have 6 consecutive flips the same?)

9. Some possible further work: Explain the sequence

$$1, \frac{1+\sqrt{5}}{2} \approx 1.618033\ldots, \frac{1+\sqrt[3]{(19-3\sqrt{33})}+\sqrt[3]{(19+3\sqrt{33})}}{3} \approx 1.83929, \ldots$$

and why does approximately 1.92756 come next? And what's the limit of this sequence?

In a game of Flipstring, you name a sequence of three flips that you would like to see, and then your opponent names a sequence of three flips. You flip a fair coin until one of those sequences occurs on any three consecutive flips. For example, if you choose HHH and your opponent chooses HTH, then after HTTHHTH, your opponent would win.

10. Out of three flips, how likely is HHH? How likely is HTH?

11. In the example game, guess the probability that the HHH player will win.

12. Out of four flips, how likely is HHH to have occurred at least once? How likely is HTH to have occurred at least once? Is it possible for both to have occurred? Is it possible for one to have occurred more than once?

13. What is the shortest sequence of flips in which both HHH and HTH occur?

14. Use your answers from the Fibonacci Flips activities to compute the probability that the HHH player has not yet seen that sequence within the first 10 flips. Can you modify your work in that problem to find the probability that the HTH player has not yet seen that sequence in 10 flips?

15. What is the probability that the HHH player wins?

16. If you go first, which string of three flips should you choose? What is the best strategy for the second player? If both players follow their best strategies, what is the probability of the first player winning?