Julia Robinson Mathematics Festival

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Euler Phi

1. List all the positive fractions with denominator 10 and numerator less than or equal to 10. How many are there?

2. Now simplify them. What denominators occur in the simplified fractions? How many fractions have each denominator?

3. Repeat the previous two problems, using denominators of 12, then 20, then 30.

4. In each case, count how many fractions are unsimplifiable, because the numerator and denominator have no common factor. This count is called the "Euler phi function", after the famous mathematician Leonhard Euler whose 300th birthday was one week ago. So for instance, $\phi(24) = 8$ because the fractions 1/24, 5/24, 7/24, 11/24, 13/24, 17/24, 19/24, and 23/24 are the only unsimplifiable ones. Organize the information you have collected into a table of values of $\phi(n)$. Hint: From your work, you know $\phi(n)$ for many values of n in addition to 10, 12, 20, and 30.

5. Based on your answers to the previous problems, you can write a formula that gives each value of n (10, 12, 20, and 30) as the sum of values of $\phi(d)$ for various values of d. Explain this formula.

6. Make a list of the value of $\phi(n)$ at least for all n from 1 through 30. By convention 1/1 is included in the list of fractions so $\phi(1) = 1$. For how many other values of n is $\phi(n)$ an odd number? Explain why.

7. Make an observation about $\phi(p)$ where p is prime. Prove your answer.

8. Experiment with numbers like 2, 4, 8, 16 ... and then 3, 9, 27, 81, ... and then 5, 25, 125, ... to determine a pattern for how many fractions will be irreducible with each of those denominators. Explain your pattern.

9. Now try numbers like 6, 12, 18, 24, 36, ... that have only 2s and 3s in their prime factorizations. Explain your pattern. You might like to visit the PIE table!

10. Is it always true that $\phi(mn) = \phi(m)\ \phi(n)$? If not, what are the exceptions? When does the rule work?

11. Determine a general formula for $\phi(n)$ in terms of the prime factorization of n.

Here, like the computer scientists, we'll say that to write a number "mod *m*" means to take the remainder when dividing by *m*, and thus to write it as a number 0 through $m-1$.

12. Write all the powers of 2, mod 3. Then try mod 5, and mod 9. Do you notice a relationship between this and the work with the ϕ function that you've done?

13. What happens differently when you look at powers of 2 mod 7?

14. Now try the powers of 3, mod 4. Try them mod 5. Mod 7. Mod 8. Mod 10. What do you discover? What happens differently if you try mod 6 or mod 9?

15. Experiment using the "irreducible" numerators. So, for example, look at the powers of 1, 5, 7, 11, 13, 17, 19, and 23, mod 24. Or, look at the powers of 1, 3, 5, and 7, mod 8. Are there shortcuts you can use in calculating those powers?

16. Use what you have learned to compute the last digit of 3^{401}. If that's too easy, then find the last two digits, or the last three digits. What is special about the choice of 401 for the exponent here?

17. What is the last digit of 7^{2007}? How about the last two digits?