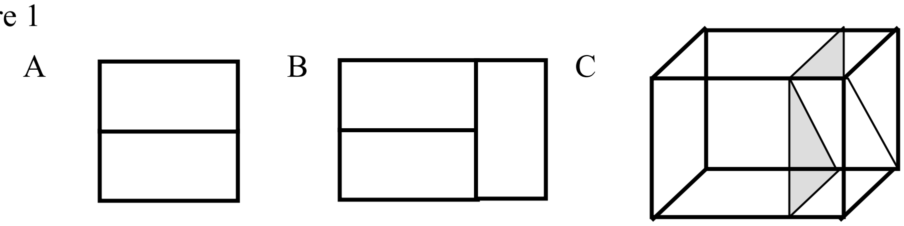
Dominoes and Rectangles

A **dissection** of a polygon is a decomposition of the polygon into finitely many polygons (called **pieces**). Similarly, in three dimensions, a dissection of a polyhedron is a decomposition of the polyhedron into finitely many polyhedrons.

In Figure 1, the 2x2 square A is dissected into two 2x1 rectangles. The 3x2 rectangle B is dissected into three 2x1 rectangles. The rectangular prism C is dissected into cube and two triangular prisms.

Figure 1

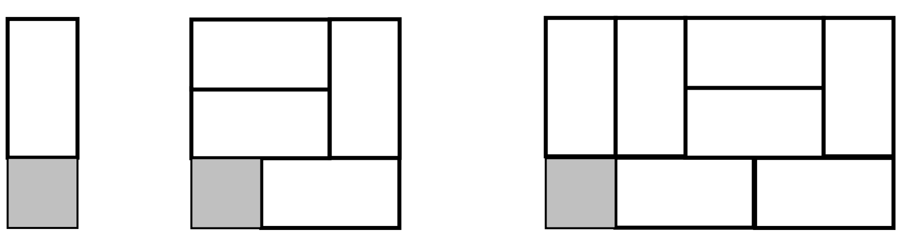


Domino Dissections

We will call a 2x1 rectangle a **domino**.

1. There is one trivial way to dissect a 2x1 rectangle into dominos. How many ways are there to dissect a 2x2 rectangle into dominos? A 2x3 rectangle into dominos? A 2x4 rectangle into dominos?
2. Find a pattern in the number of ways to dissect 2xN rectangles into dominos. Write a formula that enables you to compute the number of ways to dissect a 2xN rectangle into dominos for any N. Explain why the pattern and formula work.
3. As you found in problem 1, there are three ways to dissect a 3x2 rectangle into dominos. How many ways are there to dissect a 3x4 rectangle into dominos?
4. It is not possible to dissect a 3x1 rectangle into dominos. Likewise it is not possible to dissect 3x3, 3x5, 3x7,… rectangles into dominos. Suppose we remove a single 1x1 square from the lower left corner of these rectangles (we'll call these shapes the **3x1-1**, **3x3-1**, **3x5-1**,…). These shapes can be dissected into dominos. Figure 2 shows, for example, how to dissect the 3x1-1 shape, 3x3-1 shape, and 3x5-1 shape into dominos. How many ways are there to dissect a 3x3-1 rectangle into dominos? A 3x5-1 rectangle into dominos?

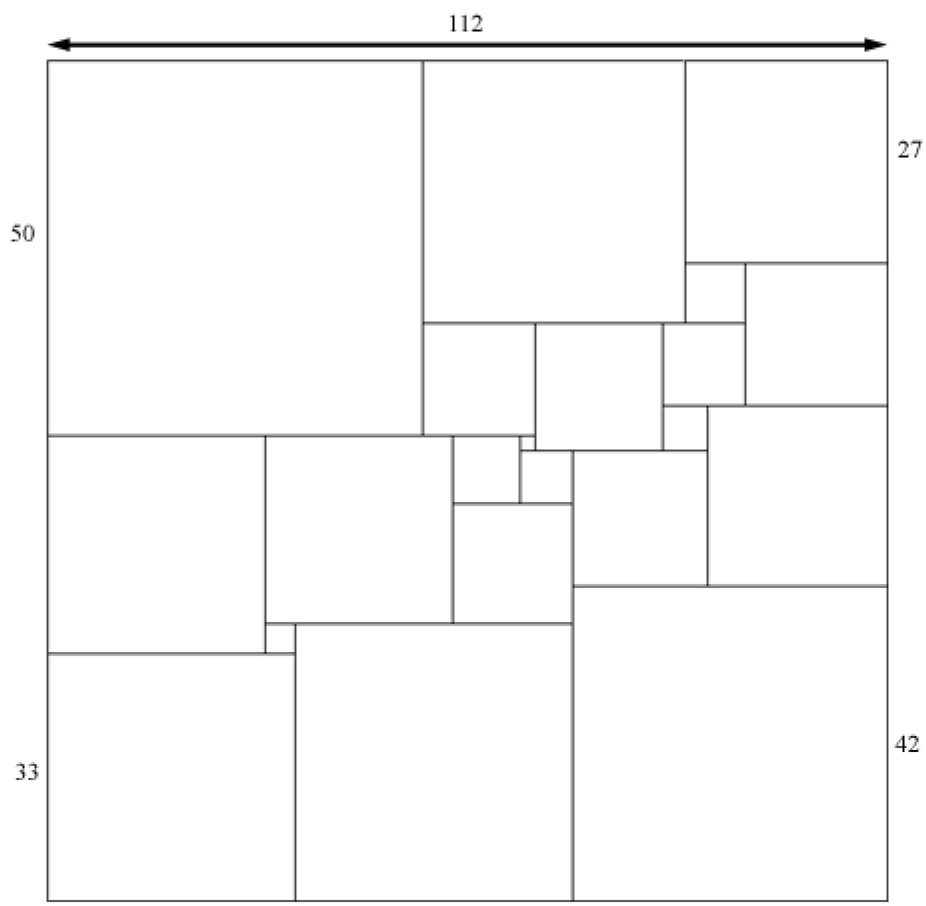
Figure 2



5. Use your answers to problems 3 and 4 to find the number of ways to dissect a 3x6 rectangle into dominos. Write a formula that enables you to compute the number ways to dissect a 3xN rectangle into dominos for any N?

Square Dissections

6. Suppose we wish to dissect a square into pieces where all the pieces are themselves squares. Call such a dissection a **square-square dissection**. There is a (trivial) square-square dissection into one piece, but there is no square-square dissection into two pieces. Is there a square-square dissection into three pieces? Into four pieces? Into five pieces?
7. What are the possible numbers of pieces of a square-square dissection?
8. A square that is dissected into smaller squares all of which are different size is called a **perfect square dissection**. The fewest number of pieces in a perfect square dissection is 21. The figure below shows a 21-piece perfect square dissection. The sizes of the square and the four corner pieces are shown in the figure. Find the sizes of 17 other squares in the dissection.



Three-Dimensional Dissections

9. Call a 2x2x1 rectangular solid a **quad**. How many ways are there to dissect 2x2x2 rectangular prism into quads? How many ways are there to dissect a 2x2x3 rectangular prism into quads? A 2x2x4 rectangular prism into quads? A 2x2x10 rectangular prism into quads?
10. Write a formula that enables you to compute the number ways to dissect a 2x2xN rectangular prism into quads for any N?
11. Suppose we wish to dissect a cube into pieces where all the pieces are themselves cubes. Call such a dissection a **cube-cube dissection**. There is a (trivial) cube-cube dissection into one piece, but there is no cube-cube dissection into two pieces. Is there a cube-cube dissection into three pieces? Into four pieces? What is the smallest number of pieces (greater than one) that a cube-cube dissection can have?
12. What are the possible numbers of pieces of a cube-cube dissection?
13. Unlike the perfect square dissection, there is no "perfect cube dissection", i.e. it is impossible to dissect a cube into smaller cubes, no two of which are the same size. Explain why. (Hint: suppose such a perfect cube exists and consider all the cubes on one of its faces.)