Julia Robinson Mathematics Festival

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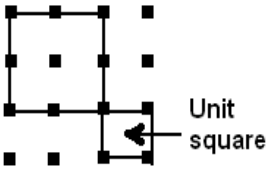
Count the squares, cubes, or hypercubes

In a square grid of dots, you can draw by connecting any two dots with a line. Those dots lie on rows and columns, such as in the drawing.

You have a certain number of unit squares in a plane. In this case, 9. You can form bigger quadrilaterals by joining 4 vertices of any of the unit squares.



1) You have a total of 9 squares, arranged to form a 3 x 3 large square, like in the picture. How many 2 x 2 squares can be formed by joining any 4 vertices of any of the grid, so that the sides of the square are parallel to the lines of the grid?

2) How many 2 x 2 squares can be formed from a 4 x 4 grid? And 3 x 3 squares?

3) How many squares of any size can be formed in the 4 x 4 grid??

Let's now think of an 8 x 8 large square.

4) How many 2 x 2 squares are there?

5) How many squares of any size can be formed from the 64 unit squares?

Now, say that the sides of the squares that are formed by joining the vertices may or may not be parallel to the sides of the large 4 x 4 square.

6) How many squares with sides of size less than 2 can you find now? (smaller than 2 x 2 squares)

7) What about squares of any size? Can you think of a rigorous way to count them? What if the grid was 8 x 8?

8) Try to compare the number of squares of each size for when the sides have to be parallel to the large square, to the numbers you have just calculated. What can you infer? Do you notice anything particular? What if the grid was n x n?

Consider a new condition: the quadrilaterals don't necessarily have to be squares; they can also be rectangles (for example, of sizes 2 x 1, 3 x 1, 3 x 2, etc).

9) Let's say first that their sides have to be parallel with the sides of the large square. How many rectangles can you find in the 4 x 4 large square? (Note: squares are particular rectangles; count them, too)

10) What if their sides are not necessarily parallel to the large square, what's the number of rectangles in the 4 x 4 square?

11) Try to think of a way to count the rectangles, too. Remember that you need all four angles to be 90º angles for the quadrilateral to be a rectangle. Can you figure out the number of rectangles in an 8 x 8 grid?

Okay. Enough with the 2D grid, let's go to 3D! Assume now that you have a 8 x 8 x 8 large cube, formed of 8^3 arranged unit cubes. First, let's consider just forming cubes with faces parallel to the large cube.

12) Remembering the first couple of questions, how many cubes of size 2 x 2 x 2 can you find in the large cube?

13) How about cubes of any size?

Now, let's say that the 3D shapes don't have to be cubes only, but can be rectangular boxes, too. Remember that a rectangular box has 90º angles.

14) How many rectangular boxes of size 2 x 1 x 1, with faces parallel to the large cube, can you count? (Note: Because of symmetry, a 2 x 1 x 1 box can be rotated around to look like a 1 x 2 x 1, or a 1 x 1 x 2, so we'll consider those, too)

15) How many rectangular boxes of any size, with faces parallel to the faces of the large cube, can you find?

16) How do you count them? Do you notice a rule? Try to generalize for a n x n x n large cube!

Think about the problem if the faces of the cubes aren't necessarily parallel to the large cube.

17) How many cubes with the size of the sides less than 2 can you find now?

18) What about cubes of any size, how many are there of those?

There's room for further questions, too. Try to answer these:

19) What if we can form squares in the 8 x 8 x 8 cube, instead of smaller cubes. How many squares of any size are there?

20) How about rectangles? Look back to the 2D numbers and correlate them with your answers here. What can you say?

21) Can you think of any other questions? Can you generalize even further?