Julia Robinson Mathematics Festival

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Coloring Cubes, or Painting Permutations

Part 1: Paint, then cut.

We'll take a cube whose edges are n units long; paint its surface completely; cut it into unit cubes, whose edges are 1 unit long; and then ask how many unit cubes are painted in which way. Clearly, if you start with a 1-unit cube, you end up with 1 unit cube painted on all 6 sides, and that's it.

1. If you start with a 2-unit cube, how many of the resulting unit cubes are completely unpainted? Painted on just 1 side? Two sides? Three sides?

2. Starting with a 3-unit cube, answer the same questions.

3. Repeat for a 4-unit cube.

4. Repeat for an n unit cube. Can you find the pattern?

Part 2: Cut, then paint.

Now we're going to first cut the cube into unit cubes, paint them, and then put them back together. But there's a little catch! We want to paint it with several different colors, in fact as many as possible, so that it can be reassembled to make a cube whose outside is all one color. With a 1-unit cube, this problem is a bit too easy. Paint the cube one color, and it's already back together.

5. With a 2-unit cube, you can cut it into 8 1-unit cubes, and paint them with two colors in such a way that you could put them back together into a 2-unit cube of either color. How should you paint them?

6. With a 3-unit cube, cut into 27 1-unit cubes, can you paint them with three colors in such a way that they can be put back together into a 3-unit cube of any of the three colors? If so, how? If not, why not?

7. Does this generalize? Starting with an n-unit cube, can you cut it into 1-unit cubes and paint them with n colors in such a way that they can be reassembled into an n-unit cube of any one of the colors?

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For TABLE LEADERS, here are some extra problems you might try with kids who find the above too easy.

Part 3: Weird cuts

Perhaps it's a bit too easy to ask how many unit cubes are cut if you drill an infinitesimally thick hole through from one corner to the diagonally opposite corner of an n-unit cube. The answer, of course, is n. So instead:

8. When the n-unit cube is cut in half by a plane perpendicular to that long diagonal, how many cubes are cut?

9. If the regions uncovered by the plane cut are painted, what is the shape of the painted region?

10. For any one unit cube, what is the least positive painted area possible?