Julia Robinson Mathematics Festival

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# Change for a Dollar

Thanks to George Polya's classic *How to Solve It* and to Bill Gosper for the ideas for several of these problems.

Our first problem is to figure out how many ways there are to make change for a dollar.

1. How many ways are there to make change for a dollar? No, really, think about it before moving on to the next problem. Even if you don't have a method, at least list a few ways, make a reasonable estimate with a reason for your estimate, and so on.

2. OK, let's simplify the problem. How many ways are there to make change for a dollar if the only coins in the world are pennies?

3. Now what if there are nickels and pennies?

4. Can you generalize: how many ways are there to make $n$ cents using only nickels and pennies?

5. Now what if there are dimes, nickels, and pennies? It might help to make an organized table, of the ways of making each amount up to a dollar (maybe in multiples of 5 cents) using dimes, nickels and pennies.

6. Now add in quarters. Do you want to allow half dollars? Dollar coins?

7. Really there are at least two possible answers to the original question. In one of them, if you get two dimes and a nickel as change for a quarter, that's it. In the other, there are three ways to get two dimes and a nickel as your change: DDN, DND, and NDD. In other words, you care about it what order you pass the coins back. Which of these questions did you answer? What would you have to change to produce the other answer?

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Now let's turn to a different problem. There's a famous theorem (which we're not going to prove) that every positive integer is the sum of four squares. For example, you can write $7 = 2^2 + 1^2 + 1^2 + 1^2$; $25 = 5^2 + 0^2 + 0^2 + 0^2 = 4^2 + 3^2 + 0^2 + 0^2 = 4^2 + 2^2 + 2^2 + 1^2$; and so on. We're going to explore how many different ways a given number can be written as the sum of four squares. And keep that last question in mind: how will you count depending on whether the order matters (so the above representation for 7 counts as 4 ways) or doesn't matter (so there's just one way to write 7 as the sum of four squares).

8. Make a table showing how many ways there are to write each number as the sum of 1 square. (Yes, a lot of numbers can be written in zero ways!)

9. In the next row of your table, show how many ways there are to write each number as the sum of 2 squares. Is there a pattern for which numbers can and cannot be written as the sum of 2 squares?

10. Now try 3 squares.

11. Finally, finish your table with 4 squares. You can see there are no more zeros in your table; can you prove there will never be?

12. Now try again, answering the other question (does order matter, or not?)

Now we'll turn our attention to triangular numbers: 0, 1, 3, 6, 10, 15, 21, …

13. Make a table showing the number of ways to write each number as the sum of four triangular numbers. As a check, your two answers for 7 should be: if order doesn't matter, 2, namely 6+1+0+0 and 3+3+1+0, and if order does matter, 24.

14. Now it appears that three triangular numbers will suffice: that is, there are lots of numbers like 5 that cannot be written as the sum of two triangular numbers, but it looks like every number can be written as the sum of three triangular numbers. Can you prove it?

15. There's an alternative way of calculating the number of (ordered) ways of writing a given number as the sum of four triangular numbers. Can you find it?

16. What other kinds of problems (besides dollars, sums of squares, and sums of triangulars) can you find that can be addressed by similar methods to these?